Balanced Three-Phase Circuits

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Structure

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Balanced Three-Phase Voltages

a-phase voltage
b-phase voltage
c-phase voltage

\[
\begin{align*}
V_a &= V_m / 0^\circ, \\
V_b &= V_m / -120^\circ, \\
V_c &= V_m / +120^\circ,
\end{align*}
\]

abc (or positive) phase sequence

\[
\begin{align*}
V_a &= V_m / 0^\circ, \\
V_b &= V_m / +120^\circ, \\
V_c &= V_m / -120^\circ.
\end{align*}
\]

acb (or negative) phase sequence
The fact that a three-phase circuit can have one of two phase sequences must be taken into account whenever two such circuits operate in parallel. The circuits can operate in parallel only if they have the same phase sequence.

*If we know the phase sequence and one voltage in the set, we know the entire set.*

From Eqs. 11.1 and 11.2, we have

\[
V_a + V_b + V_c = 0
\]

\[
v_a + v_b + v_c = 0
\]
Three-Phase Voltage Sources

a. Rotation of the electromagnet induces a sinusoidal voltage in each winding.

b. The phase windings are designed so that the sinusoidal voltages induced in them are equal in amplitude and out of phase with each other by 120°.

c. The phase windings are stationary with respect to the rotating electromagnet, so the frequency of the voltage induced in each winding is the same.
There are two ways of interconnecting the separate phase windings to form a three-phase source:

Y-connected

Δ-connected

The neutral terminal **may or may not** be available for external connections.

Here, the model consists solely of **ideal voltage sources**, because the impedance of each phase winding is so small (compared with other impedances in the circuit) that we need not account for it in modeling the generator.

**Electric Circuits**
If the impedance of each phase winding is not negligible, we place the *winding impedance in series* with an Ideal sinusoidal voltage source:

**Y-connected**

\[ R_w \quad jX_w \]

\[ V_a \quad \quad V_b \quad \quad V_c \]

\[ jX_w \]

\[ R_w \]

**Δ-connected**

\[ R_w \quad jX_w \]

\[ V_a \]

\[ V_b \quad \quad V_c \]

\[ jX_w \]

\[ R_w \]

**Four kinds of circuits**

<table>
<thead>
<tr>
<th>Source</th>
<th>Load</th>
</tr>
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<tbody>
<tr>
<td>Y</td>
<td>Y</td>
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<tr>
<td>Y</td>
<td>Δ</td>
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<tr>
<td>Δ</td>
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**Problems**
Analysis of the Wye-Wye Circuit

Node-voltage equation:

\[ \frac{V_N}{Z_0} + \frac{V_N - V_{a'n}}{Z_A + Z_{la} + Z_{ga}} + \frac{V_N - V_{b'n}}{Z_B + Z_{lb} + Z_{gb}} + \frac{V_N - V_{c'n}}{Z_C + Z_{lc} + Z_{gc}} = 0 \]

Electric Circuits

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Balanced Three-phase Circuit

a. The voltage sources form a set of balanced three-phase voltages. This means that $V_{a'n}$, $V_{b'n}$, and $V_{c'n}$ are a set of balanced three-phase voltages.

b. The impedance of each phase of the voltage source is the same. This means that $Z_{ga} = Z_{gb} = Z_{gc}$.

c. The impedance of each line (or phase) conductor is the same. This means that $Z_{1a} = Z_{1b} = Z_{1c}$.

d. The impedance of each phase of the load is the same. This means that $Z_A = Z_B = Z_Q$.

*There is no restriction on the impedance of a neutral conductor; its value has no effect on whether the system is balanced.*
Then, for a balanced three-phase circuit, Eq. 11.5 can be rewritten as:

\[ V_N \left( \frac{1}{Z_0} + \frac{3}{Z_\phi} \right) = \frac{V_{a'n} + V_{b'n} + V_{c'n}}{Z_\phi} \]

where

\[ Z_\phi = Z_A + Z_{1a} + Z_{ga} = Z_B + Z_{1b} + Z_{gb} = Z_C + Z_{1c} + Z_{gc} \]

There is no difference in potential between the source neutral, \( n \), and the load neutral, \( N \); consequently, the current in the neutral conductor is zero. Hence we may either remove the neutral conductor from a balanced Y-Y configuration \( (I_0 = 0) \) or replace it with a perfect short circuit between the nodes \( n \) and \( N \) \( (V_N = 0) \).

The current in each line is equal in amplitude and frequency and is 120° out of phase with the other two line currents. Thus, if we calculate the current \( I_{aA} \) and we know the phase sequence, we have a shortcut for finding \( I_{bB} \) and \( I_{cC} \).

\[ I_{aA} = \frac{V_{a'n} - V_N}{Z_A + Z_{1a} + Z_{ga}} = \frac{V_{a'n}}{Z_\phi} \]
\[ I_{bB} = \frac{V_{b'n} - V_N}{Z_B + Z_{1b} + Z_{gb}} = \frac{V_{b'n}}{Z_\phi} \]
\[ I_{cC} = \frac{V_{c'n} - V_N}{Z_C + Z_{1c} + Z_{gc}} = \frac{V_{c'n}}{Z_\phi} \]
Single-phase Equivalent Circuit

Once we know the line current in the figure, calculating any voltages of interest is relatively simple.

The current in the neutral conductor this circuit is $I_{aA}$, which is not the same as the current in the neutral conductor of the balanced three-phase circuit, which is

$$I_{\omega} = I_{aA} + I_{bB} + I_{cC} = 0$$

Balanced
Of particular interest is the relationship between the line-to-line voltages and the line-to-neutral voltages.

\[ V_{AB} = V_{AN} - V_{BN} \]
\[ V_{BC} = V_{BN} - V_{CN} \]
\[ V_{CA} = V_{CN} - V_{AN} \]

We assume a positive, or abc, sequence and the line-to-neutral voltage of the a-phase as the reference:

\[ V_{AN} = V_\phi \angle 0^\circ \]
\[ V_{CN} = V_\phi \angle +120^\circ \]
\[ V_{BN} = V_\phi \angle -120^\circ \]

\[ V_{AB} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ = \sqrt{3}V_\phi \angle 30^\circ \]
\[ V_{BC} = V_\phi \angle -120^\circ - V_\phi \angle 120^\circ = \sqrt{3}V_\phi \angle -90^\circ \]
\[ V_{CA} = V_\phi \angle 120^\circ - V_\phi \angle 0^\circ = \sqrt{3}V_\phi \angle 150^\circ \]
1. The magnitude of the line-to-line voltage is $\sqrt{3}$ times the magnitude of the line-to-neutral voltage.
2. The line-to-line voltages form a balanced three-phase set of voltages.
3. The set of line-to-line voltages leads the set of line-to-neutral voltages by $30^\circ$.

**Line voltage:** the voltage across any pair of lines  
**Phase voltage:** the voltage across a single phase  
**Line current:** the current in a single line  
**Phase current:** current in a single phase

Observe that *in a $\Delta$ connection, line voltage and phase voltage are identical*,  
and *in a $Y$ connection, line current and phase current are identical*. 

**Electric Circuits**
Some Notes

• Because three-phase systems are designed to handle large blocks of electric power, all voltage and current specifications are given as rms values.

• When voltage ratings are given, they refer specifically to the rating of the line voltage. Thus when a three-phase transmission line is rated at 345 kV, the nominal value of the rms line-to-line voltage is 345,000 V.

• In this chapter we express all voltages and currents as rms values.

• Finally, the Greek letter phi (ϕ) is widely used in the literature to denote a per-phase quantity. Thus \( V_\phi \), \( I_\phi \), \( Z_\phi \), \( P_\phi \) and \( Q_\phi \) are interpreted as voltage/phase, current/phase, impedance/phase, power/phase, and reactive power/phase, respectively.
Example #1

A balanced three-phase Y-connected generator with positive sequence has an impedance of $0.2 + j0.5\ \Omega/\phi$ and an internal voltage of $120\ V/\phi$. The generator feeds a balanced three-phase Y-connected load having an impedance of $39 + j28\ \Omega/\phi$. The impedance of the line connecting the generator to the load is $0.8 + j1.5\ \Omega/\phi$. The a-phase internal voltage of the generator is specified as the reference phasor.

• Construct the a-phase equivalent circuit of the system.
• Calculate the three line currents $I_{aA}$, $I_{bB}$, and $I_{cC}$.

a) Calculate the three phase voltages at the load. $V_{AN}$, $V_{BN}$, and $V_{CN}$.

b) Calculate the line voltages $V_{AB}$, $V_{BC}$, and $V_{CA}$ at the terminals of the load.

c) Calculate the phase voltages at the terminals of the generator, $V_{an}$, $V_{bn}$, and $V_{cn}$.

d) Calculate the line voltages $V_{ab}$, $V_{bc}$, and $V_{ca}$ at the terminals of the generator.
Solution for Example #1

b) \[ \mathbf{I}_{aA} = \frac{120^{\circ}}{0^\circ} \left( \frac{0.2 + 0.8 + 39}{0.8 + 1.5 + 28} \right) \]
\[ = \frac{120^{\circ}}{40 + j30} = 2.4^{\circ} -36.87^\circ A \]
\[ \mathbf{I}_{bB} = 2.4^{\circ} -156.87^\circ A \quad \mathbf{I}_{cC} = 2.4^{\circ} 83.13^\circ A \]

c) \[ \mathbf{V}_{AN} = (39 + j28)(2.4^{\circ} -36.87^\circ) = 115.22^{\circ} -1.19^\circ V \]
\[ \mathbf{V}_{BN} = 115.22^{\circ} -121.19^\circ V \quad \mathbf{V}_{CN} = 115.22^{\circ} 118.81^\circ V \]

d) \[ \mathbf{V}_{AB} = (\sqrt{3}^{\circ} /30^\circ) \mathbf{V}_{AN} = 199.58^{\circ} 28.81^\circ V \]
\[ \mathbf{V}_{BC} = 199.58^{\circ} -91.19^\circ V \quad \mathbf{V}_{CA} = 199.58^{\circ} 148.81^\circ V \]

Electric Circuits
e) 

\[ V_{an} = 120 - (0.2 + j0.5)(2.4 \angle -36.87^\circ) \]
\[ = 120 - 1.29 \angle 31.33^\circ = 118.90 - j0.67 = 118.90 \angle -0.32^\circ \text{ V} \]

\[ V_{bn} = 118.90 \angle -120.32^\circ \text{ V} \quad V_{cn} = 118.90 \angle 119.68^\circ \text{ V} \]

f) 

\[ V_{ab} = (\sqrt{3} \angle 30^\circ) V_{an} = 205.94 \angle 29.68^\circ \text{ V} \]

\[ V_{bc} = 205.94 \angle -90.32^\circ \text{ V} \quad V_{ca} = 205.94 \angle 149.68^\circ \text{ V} \]
Analysis of the Wye-Delta Circuit

If the load in a three-phase circuit is connected in a delta, it can be transformed into a wye by using the delta-to-wye transformation discussed in Section 9.6. *When the load is balanced, the impedance of each leg of the wye is one third the impedance of each leg of the delta, or*

For *balanced* load, we have

\[ Z_Y = \frac{Z_\Delta}{3} \]

See Eqs. 9.51-9.53

**Electric Circuits**
We use this circuit to calculate the line currents, and we then use the line currents to find the currents in each leg of the original \( \Delta \) load. The relationship between the line currents and the currents in each leg of the delta can be derived using the circuit shown in the Figure.

To demonstrate the relationship between the phase currents and line currents, we assume a positive phase sequence and let \( I_\phi \) represent the magnitude of the phase current. Then

\[
\begin{align*}
I_{AB} &= I_\phi / 0^\circ \\
I_{BC} &= I_\phi / -120^\circ \\
I_{CA} &= I_\phi / 120^\circ
\end{align*}
\]
The magnitude of the line currents is $\sqrt{3}$ times the magnitude of the phase currents and that the set of line currents lags the set of phase currents by $30^\circ$. 

**Electric Circuits**
Example #2

The Y-connected source in Example #1 (p. 427) feeds a Δ-connected load through a distribution line having an impedance of $0.3 + j0.9 \ \Omega / \phi$. The load impedance is $118.5 + j85.8 \ \Omega / \phi$. Use the a-phase internal voltage of the generator as the reference.

• Construct a single-phase equivalent circuit of the three-phase system.

• Calculate the line currents $I_{aA}$, $I_{bB}$, and $I_{cC}$.

a) Calculate the phase voltages at the load terminals.

b) Calculate the phase currents of the load.

c) Calculate the line voltages at the source terminals.
Solution for Example #2

a) The load impedance of the Y equivalent is:

\[
\frac{118.5 + j85.8}{3} = 39.5 + j28.6 \, \Omega/\phi.
\]

b) \[I_{aA} = \frac{120/0^\circ}{(0.2 + 0.3 + 39.5) + j(0.5 + 0.9 + 28.6)} = \frac{120/0^\circ}{40 + j30} = 2.4/\overline{-36.87}^\circ \, A\]

\[I_{bB} = 2.4/\overline{-156.87}^\circ \, A \quad I_{cC} = 2.4/83.13^\circ \, A\]

c) \[V_{AN} = (39.5 + j28.6)(2.4/\overline{-36.87}^\circ) = 202.72/\overline{29.04}^\circ \, V\]

\[V_{BC} = 202.72/\overline{-90.96}^\circ \, V \quad V_{CA} = 202.72/\overline{149.04}^\circ \, V\]
d) The phase currents of the load may be calculated directly from the line currents:

\[ I_{AB} = \left( \frac{1}{\sqrt{3}} \angle 30^\circ \right) I_{aA} = 1.39 \angle -6.87^\circ \ A \]

\[ I_{BC} = 1.39 \angle 126.87^\circ \ A \quad I_{CA} = 1.39 \angle 113.13^\circ \ A \]

e) 

\[ V_{an} = (39.8 + j29.5)(2.4 \angle -36.87^\circ) = 118.90 \angle -0.32^\circ \ V \]

\[ V_{ab} = (\sqrt{3} \angle 30^\circ) V_{an} = 205.94 \angle 29.68^\circ \ V \]

\[ V_{bc} = 205.94 \angle -90.32^\circ \ V \quad V_{ca} = 205.94 \angle 149.68^\circ \ V . \]
Power Calculations in Balanced Three-Phase Circuits

Average power in a balanced Wye load

\[
\begin{align*}
P_A &= |V_{AN}| |I_{aA}| \cos (\theta_{vA} - \theta_{iA}) \\
P_B &= |V_{BN}| |I_{bB}| \cos (\theta_{vB} - \theta_{iB}) \\
P_C &= |V_{CN}| |I_{cC}| \cos (\theta_{vC} - \theta_{iC})
\end{align*}
\]
If we denote

\[ V_\phi = |V_{AN}| = |V_{BN}| = |V_{CN}| \]

\[ I_\phi = |I_{aA}| = |I_{bB}| = |I_{cC}| \]

\[ \theta_\phi = \theta_{vA} - \theta_{iA} = \theta_{vB} - \theta_{iB} = \theta_{vC} - \theta_{iC} \]

We will have

\[ P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos \theta_\phi \]

\[ P_T = 3P_\phi = 3V_\phi I_\phi \cos \theta_\phi \]

\[ P_T = 3\left(\frac{V_L}{\sqrt{3}}\right)I_L \cos \theta_\phi = \sqrt{3}V_L I_L \cos \theta_\phi \]

\[ V_L \text{ and } I_L \text{ are the rms magnitudes of the line voltage and current} \]
Complex Power in a Balanced Wye Load

Similarly, we can get the following reactive power:

\[ Q_\phi = V_\phi I_\phi \sin \theta_\phi \]
\[ Q_T = 3Q_\phi = \sqrt{3}V_L I_L \sin \theta_\phi \]

For a balanced load

\[ S_\phi = V_{AN} I_{aA}^* = V_{BN} I_{bB}^* = V_{CN} I_{cC}^* = V_\phi I_\phi^* \]

Thus, in general

\[ S_\phi = P_\phi + jQ_\phi = V_\phi I_\phi^* \]
\[ S_T = 3S_\phi = \sqrt{3}V_L I_L \angle \theta_\phi \]
The power associated with each phase is:

\[ P_A = |V_{AB}||I_{AB}| \cos (\theta_{vAB} - \theta_{iAB}) \]
\[ P_B = |V_{BC}||I_{BC}| \cos (\theta_{vBC} - \theta_{iBC}) \]
\[ P_C = |V_{CA}||I_{CA}| \cos (\theta_{vCA} - \theta_{iCA}) \]

For a balanced load,

\[ |I_{AB}| = |I_{BC}| = |I_{CA}| = I_\phi \]
\[ |V_{AB}| = |V_{BC}| = |V_{CA}| = V_\phi \]
\[ \theta_{vAB} - \theta_{iAB} = \theta_{vBC} - \theta_{iBC} = \theta_{vCA} - \theta_{iCA} = \theta_\phi \]

Thus, we have

\[ P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos \theta_\phi \]

**Electric Circuits**
Comparing Eq. 11.48 and Eq. 11.34, we can see that, in a balanced load, regardless of whether it is Y- or Δ-connected, the average power per phase is equal to the product of the rms magnitude of the phase voltage, the rms magnitude of the phase current, and the cosine of the angle between the phase voltage and current.

The expressions for the total power delivered, reactive power and complex power also have the same form as those developed for the Y load:

\[
P_T = 3P_\phi = 3V_\phi I_\phi \cos \theta_\phi = 3V_L \left( \frac{I_L}{\sqrt{3}} \right) \cos \theta_\phi = \sqrt{3}V_L I_L \cos \theta_\phi
\]

\[
Q_\phi = V_\phi I_\phi \sin \theta_\phi \quad Q_T = 3Q_\phi = 3V_\phi I_\phi \sin \theta_\phi
\]

\[
S_\phi = P_\phi + jQ_\phi = V_\phi I_\phi^* \quad S_T = 3S_\phi = \sqrt{3}V_L I_L / \theta_\phi
\]
Instantaneous Power in Three-Phase Circuits

\[ p_A = v_{AN}i_{aA} = V_mI_m \cos \omega t \cos (\omega t - \theta_\phi) \]
\[ p_B = v_{BN}i_{bB} = V_mI_m \cos (\omega t - 120^\circ) \cos (\omega t - \theta_\phi - 120^\circ) \]
\[ p_C = v_{CN}i_{cC} = V_mI_m \cos (\omega t + 120^\circ) \cos (\omega t - \theta_\phi + 120^\circ) \]

\[ V_m = \sqrt{2}V_\phi \]
\[ I_m = \sqrt{2}I_\phi \]

\[ p_T = p_A + p_B + p_C = 1.5V_mI_m \cos \theta_\phi = 3V_\phi I_\phi \cos \theta_\phi \]

It is invariant with time!
Example #3

a) Calculate the average power per phase delivered to the Y-connected load of Example #1 (P. 427).
b) Calculate the total average power delivered to the load.
c) Calculate the total average power lost in the line.
d) Calculate the total average power lost in the generator.
e) Calculate the total number of magnetizing vars absorbed by the load.
f) Calculate the total complex power delivered by the source.
Solution for Example #3

From Example #1, $V_\phi = 115.22 \text{ V}$, $I_\phi = 2.4 \text{ A}$, and $\theta_\phi = -1.19 - (-36.87) = 35.68^\circ$. Therefore

$$P_\phi = (115.22)(2.4) \cos 35.68^\circ = 224.64 \text{ W}$$

The total average power delivered to the load is

$$P_T = 3P_\phi = 673.92 \text{ W}$$

The total power lost in the line is

$$P_{\text{line}} = 3(2.4)^2(0.8) = 13.824 \text{ W}$$

The total internal power lost in the generator is

$$P_{\text{gen}} = 3(2.4)^2(0.2) = 3.456 \text{ W}$$

The total number of magnetizing vars absorbed by the load is

$$Q_T = \sqrt{3}(199.58)(2.4) \sin 35.68^\circ = 483.84 \text{ VAR}$$

The total complex power associated with the source is

$$S_T = 3S_\phi = -3(120)(2.4) \angle 36.87^\circ = -691.20 - j518.40 \text{ VA}$$

Electric Circuits
Measuring Average Power in Three-Phase Circuits

The basic instrument used to measure power in three-phase circuits is the electrodynamometer wattmeter. It contains two coils. One coil, called the current coil, is stationary and is designed to carry a current proportional to the load current. The second coil, called the potential coil, is movable and carries a current proportional to the load voltage.
Consider a general network inside a box to which power is supplied by $n$ conducting lines. \textit{If we wish to measure the total power at the terminals of the box, we need to know $n - 1$ currents and voltages.} This follows because if we choose one terminal as a reference, there are only $n - 1$ independent voltages. Likewise, only $n - 1$ independent currents can exist in the $n$ conductors entering the box. Thus the total power is the sum of $n - 1$ product terms; that is, $p = v_1i_1 + v_2i_2 + \ldots + v_{n-1}i_{n-1}$.

Applying this general observation, we can see that for a three-conductor circuit, whether balanced or not, we need only two wattmeters to measure the total power.
\[ W_1 = |V_{AB}||I_{aA}| \cos \theta_1 = V_L I_L \cos \theta_1 \]
\[ W_2 = |V_{CB}||I_{cC}| \cos \theta_2 = V_L I_L \cos \theta_2 \]

\[ \theta_1 = \theta + 30^\circ = \theta_\phi + 30^\circ \]
\[ \theta_2 = \theta - 30^\circ = \theta_\phi - 30^\circ \]

\[ W_1 = V_L I_L \cos (\theta_\phi + 30^\circ) \]
\[ W_2 = V_L I_L \cos (\theta_\phi - 30^\circ) \]

\[ P_T = W_1 + W_2 = 2V_L I_L \cos \theta_\phi \cos 30^\circ = \sqrt{3} V_L I_L \cos \theta_\phi \]

1. If the power factor is greater than 0.5, both wattmeters read positive.
2. If the power factor equals 0.5, one wattmeter reads zero.
3. If the power factor is less than 0.5, one wattmeter reads negative.
Conclusion

• Balanced three-phase circuit
• Abc/acb phase sequence
• Y-Y/Y- Δ  circuits
• Phase/line voltage/current
• Per-phase average/reactive/complex power
• Two-Wattmeter method