Consensus of multi-agent systems in the cooperation-competition network with inherent nonlinear dynamics: A time-delayed control approach

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Abstract

In this paper, the consensus problem is investigated for a group of first-order agents in the cooperation-competition network, where agents can cooperate or even compete with each other, i.e., the elements in the coupling weight matrix of the graph can be either positive or negative. In order to solve this consensus problem, the whole network is firstly divided into two sub-networks, i.e., the cooperation sub-network and the competition sub-network, and then two kinds of time-delayed control schemes are designed in the competition sub-network. By combining the Lyapunov theory together with the synchronization manifold method, several effective sufficient conditions of consensus are provided without assuming that the interaction topology is strongly connected or contains a directed spanning tree, which means that the competition relationships could help the agents achieve consensus under the time-delayed control designed in the competition sub-network. Moreover, the results are also extended to the pure competition networks where all the elements in the weight matrices are either zeros or negative. Finally, some simulation examples are provided to validate the effectiveness of the theoretical analysis.

Keyword: Multi-agent systems; Consensus; Time-delayed control; Cooperation and competition.

I. INTRODUCTION

Over the past few years, increasing attention has been paid to the study of multi-agent systems (MAS) across many fields of science and engineering. Generally, a multi-agent system is always composed of many interconnected agents, in which agents represent individual elements with their own dynamics and links represent certain relationships between their dynamics. Applications are ubiquitous in the real world, such as the World Wide Web [1] where the web pages as agents are connected by hyperlinks, the Social Network [2] in which the agents are persons and the links represent the relationships between them, and the Gene Regulatory Network

[3] in which the genes as agents are connected by biological signals, etc.

Consensus is one of the most important issues in the multi-agent systems, which is always used to explain flocking of social animals and has been widely applied in many engineering areas such as air traffic control, wireless sensor networks and mobile robotic swarms [4]-[9]. The main objective of a consensus problem is to design an appropriate algorithm or interaction rule such that a group of agents converges to a consistent quantity of interest. Here, the algorithm or the interaction rule is usually called agreement protocol, and the consistent quantity depending on the initial states of all agents is called consensus state that may represent certain physical quantity such as attitude, position, temperature, voltage, etc.

In the past decade, researchers began to study consensus problems of multi-agent systems by considering different subsistent limitations, such as finite-time consensus [10]-[12], higher-order consensus [13]-[15], leader-following consensus [16]-[18], heterogeneous consensus [19]-[21], Adaptive consensus [22]-[24], etc. For example, Ma et al [25] considered the consensus problem of second-order multi-agent systems with sampled data, where the sufficient consensus condition was derived and the upper bound of sampling interval was estimated by adopting a novel time-dependent Lyapunov function. Ren [26] further analyzed such second-order consensus problem in four cases, where several conditions were derived based on the interaction topology. Yu et al. [27] studied some necessary and sufficient conditions for second-order consensus in directed networks containing a directed spanning tree, where they proved that both the real and imaginary parts of the eigenvalues of the communication graph's Laplacian matrix play key roles in reaching such consensus. Xiao and Wang [28] studied asynchronous consensus problem for continuous-time multi-agent systems with discontinuous information transmission by adopting nonnegative matrix analysis and graph theory, and the asynchronous consensus problem was further investigated in [29]. Li et al. [30] considered the consensus problem of multi-agent systems with a time-invariant communication topology consisting of nodes with general linear dynamics, and introduced a novel framework that can describe the consensus of multi-agent systems and the synchronization of complex dynamic networks in a unified way. Liang et al. [31] investigated a new synchronization problem for an array of 2-D coupled dynamical networks where all the agents were governed by the Fornasini-Marchesini system, and derived several sufficient conditions by adopting the energy-like quadratic function. More recently, Wen et al. [32] studied the second-order consensus problem with communication constraints where each agent is assumed to share information only with its neighbors on some disconnected time intervals, which is then solved by a novel protocol with synchronous intermittent information feedback. Liu and Zhao [33] investigated the generalized output synchronization problem for dynamical networks using the output synchronization without assuming the negative definiteness property of the coupling matrix of the network. Ma and Lu [34] studied the cluster synchronization problem of a class of general complex dynamical networks, and the network topology was assumed to be directed and weakly connected. By the pinning control scheme, some simple control criteria were proposed.

Sometimes, the coupling delays [35, 36] between agents need to be considered in real circumstance with practical reasons such as the communication congestion of the channels, the finite switching and spreading speed of the hardware and circuit implementation, the moving of the agents, etc. Lu et al. [37] proposed an adaptive scheme for the stabilization and synchronization of chaotic Lur'e systems with time-varying delay. Based on the work introduced in [18], Lin et al. [38] investigated consensus problems in networks of continuous-time agents with diverse time-delays and jointly-connected topologies, where several sufficient conditions were derived by adopting the Lyapunov–Krasovskii approach and it can be found that all the agents can reach consensus even though the communication structures among agents dynamically change over time so that the corresponding graphs may not be connected. Yu et al. [39] investigated the global synchronization problem of a generalized linearly hybrid coupled network with time-varying delay, and several effective sufficient conditions of global synchronization were obtained based on the Lyapunov function and the linear matrix inequality (LMI). Recently, Guan et al. [40] considered the consensus problem with system delay and multiple coupling delays via impulsive distributed control, and introduced the concept of control topology that describe the whole controller structure. Wang et al. [41] presented the coupled discrete-time stochastic complex network with randomly occurred nonlinearities and time delays. In their paper, several delay-dependent sufficient conditions were obtained which ensure the asymptotic synchronization in the mean square sense by employing a combination of LMI, the free-weighting matrix method and stochastic analysis theory. Qin et al. [42] studied the consensus problems for second-order agents under directed arbitrarily switching topologies with communication delay, and they proved that consensus can be reached if the delay is small enough.

However, almost all of the previous studies were only concerned with the cooperation network, i.e., the weight matrix of the network is assumed to be nonnegative, which may give rise to the problem that the network cannot fully represent the real physical object. Therefore, it needs to be further generalized in some cases. This paper focuses on studying the consensus problem in cooperation-competition networks where the entries of the corresponding weight matrices may be negative, which can further lead to the negative off-diagonal entries of the Laplacian matrices. Generally, such a cooperation-competition network can be divided into two sub-networks, i.e., the cooperation and competition sub-networks with the links having positive and negative weights, respectively. Since the competitions between agents may prevent their consensus, the situation introduced here is much more complicated than most former cases where only cooperation is considered. In this paper, a time-delayed control scheme is designed in the competition sub-network, which could overcome the negative factor and help the agents achieve consensus. Meanwhile, such control scheme is typically simple to be implemented. Based on the viewpoint of the synchronization manifold in [39, 43], several sufficient conditions of consensus are then deduced by using the Lyapunov method and the linear matrix inequality (LMI).

The rest of the paper is organized as follows. In Section II, some basic definitions in graph theory and related the mathematical preliminary results are presented. Then, the two kinds of time-delayed control schemes are described. In Section III, main analytical results are established according to different time-delayed control schemes. In Section IV, numerical simulations are implemented to demonstrate the analytic results. Finally, the paper is concluded in Section V.

II. PROBLEM FORMULATION

In this section, some basic definitions in graph theory, preliminary mathematical results, the system model, and two kinds of time-delayed control schemes are firstly introduced for subsequent use.

The mathematical notations which will be employed in this paper are presented as follows. Let R^n denote the n-dimensional real vector space, and the Euclidean norms of a vector $x \in R^n$ and a matrix $A \in R^{n\times n}$ are denoted by $\|x\| \triangleq \sqrt{x^T x}$ and $\|A\| \triangleq \sqrt{\lambda_{\max}(A^T A)}$ with $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ being the maximum and minimum eigenvalues of the matrix A, respectively. The matrix $A > 0$ or $A < 0$ denotes that A is symmetric and positive or negative definite matrix. Besides, the identity matrix of order n is denoted by I_n and the Kronecker product of matrices $A \in R^{n\times n}$ and $B \in R^{n\times n}$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix},$$

which satisfies the following properties:

$$\|I_P \otimes A\| = \|A \otimes I_N\| = \|A\|,$$

$$(A+B)\otimes C = A\otimes C + B\otimes C,$$

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD).$$

A. Topology Description

In general, information exchanges between agents in a multi-agent system can be modeled by a network or graph [4, 44]. Let $\Re = (V, \zeta, A)$ be a weighted directed network of N agents with a set of nodes $V = \{\pi_1, \pi_2, \cdots \pi_N\}$, a set of links $\zeta \subseteq V \times V$, and a weight matrix $A = [a_{ij}]$, which represents the communication topology. A link of $\Re$ is denoted by $e_{ij} = (\pi_i, \pi_j)$ which is associated with a nonzero weight, i.e. $(\pi_i, \pi_j) \in \zeta \Leftrightarrow a_{ij} \neq 0$, then the neighbor set of node π_i is denoted by $N_i = \{\pi_j \,|(\pi_j, \pi_i) \in \zeta\}$. Note that here $a_{ij} < 0$ also makes sense, which means that agent i competes with agent j, and thus prevent the consensus of the system. Hence, for every node π_i, the neighbor set could be divided into two parts, i.e., $N_i = \{\pi_j \,|a_{ij} > 0\} \cup \{\pi_j \,|a_{ij} < 0\}$ with the sets $\{\pi_j \,|a_{ij} > 0\}$ and $\{\pi_j \,|a_{ij} < 0\}$, which are denoted by N_{1i} and N_{2i} representing the cooperation and competition neighbors of π_i, respectively.

Here it is assumed that $a_{ii} = 0$ for all $i \in I$, i.e., self-loop is not allowed in the graph. The

Laplacian matrix $L=\left[l_{ij}\right]\in R^{N\times N}$ associated with the weight matrix A is defined as

$$\begin{aligned} l_{ij} &= -a_{ij}, i\neq j, \\ l_{ii} &= \sum_{j\in N_i} a_{ij}, \end{aligned} \qquad (1)$$

which ensures that $\sum_{j=1}^{N} l_{ij}=0, \forall i\in I$. Similar to the neighbor set, the Laplacian matrix L could also be divided into two parts, i.e., $L=L_1-L_2$, where L_1 and $-L_2$ are the Laplacian matrices of the cooperation and competition sub-networks, respectively.

B. Mathematic Preliminaries

Let $T(\varepsilon)$ denote the set of matrices with the sum of the elements in each row being the real number ε. Denote by M_1^N the set of matrices with N columns and each row containing exactly two nonzero elements, i.e., 1 and -1. Then the subset $M_2^N\subseteq M_1^N$ is defined as follow: if $M=[m_{ij}]\in M_2^N$, then for each pair of column indices i and j, there exist indices $j_1, j_2,\cdots, j_l$ and $p_1, p_2,\cdots, p_{l-1}$ such that $m_{p_q j_q}\neq 0$ and $m_{p_q j_{q+1}}\neq 0$ for $q=1,2,\cdots,l-1$, with $j_1=i$ and $j_l=j$.

Similar to [39, 43], the following lemma will be used in the derivations of the main results.

Lemma 1. Let $M\in M_2^N$ be a $P\times N$ matrix and H be an $N\times N$ matrix. Then there exists an $N\times P$ matrix G_M such that $MH=\hat{H}M$ with $\hat{H}=MHG_M$. In particular, if $M\in M_2^N$ is an $(N-1)\times N$ matrix, then $MG_M=I_{N-1}$.

Remark 1. The matrix M plays an important role in the analysis of consensus. In fact, let $x=(x_1^T, x_2^T,\cdots x_N^T)^T$, where $x_i\in R^n$, $i=1,\cdots,N$, according to the assumptions on M, it is found that $\left\|(M\otimes I_n)x\right\|=0$ if and only if $\left\|x_i(t)-x_j(t)\right\|=0$ for all $i,j=1,\cdots,N$. Hence, $\left(M\otimes I_n\right)x$ can be used to analyze the consensus of the multi-agent systems, which is called the synchronization manifold in [39, 43].

Lemma 2. For any real differentiable vector function $w(t)\in R^n$, scalar $r>0$, and any constant matrix $0<Q=Q^T\in R^{n\times n}$, the following inequality holds:

$$\int_{t-r}^{t}\dot{w}^T(s)Q\dot{w}(s)ds\geq\frac{1}{r}\left[w(t)-w(t-r)\right]^T Q\left[w(t)-w(t-r)\right].$$

C. System Model and Time-Delayed Control

Let us introduce the multi-agent systems with nonlinear dynamics. Consider a nonlinear multi-agent system composed of N coupled autonomous agents in the cooperation-competition network:

$$\dot{x}_i(t)=-Cx_i(t)+f(t,x_i(t))+\sum_{j\in N_i}a_{ij}(x_j(t)-x_i(t)), \quad i=1,\cdots,N, \qquad (1)$$

where $C\in R^{n\times n}$, and $f(t,x_i(t))\in R^n$ is the intrinsic dynamics of agent i. For simplification, just like most recent works on second-order consensus problems with nonlinear agent dynamics [43, 45], here it is supposed that each agent has the same intrinsic dynamics. According to the decomposition of the cooperation-competition network, Eq. (1) can be rewritten as:

$$\dot{x}_i(t) = -Cx_i(t) + f(t, x_i(t)) + \sum_{j \in N_{1i}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j \in N_{2i}} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \cdots, N . \quad (2)$$

Denote $F(t,x) = \left[f^T(t, x_1), \cdots, f^T(t, x_N) \right]^T$. Then, by utilizing vector notations, Eq. (2) can be rewritten with a compact form as follows:

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) - (L_1 \otimes I_n)x(t) + (L_2 \otimes I_n)x(t) . \quad (3)$$

Due to the competition relationships between agents, Eq. (3) could not directly achieve consensus. Therefore, the extra controller needs to be adopted in order to overcome the negative factor and help the agents achieve consensus. In this paper, two kinds of time-delayed control schemes are designed in the competition sub-network, i.e.,

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) - (L_1 \otimes I_n)x(t) + e^{-\alpha\tau}(L_2 \otimes I_n)x(t - \tau) , \quad (4)$$

and

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) - (L_1 \otimes I_n)x(t) + (L_2 \otimes I_n)\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds , \quad (5)$$

respectively, where $\alpha > 0$ is a decay parameter and $\tau > 0$ is a control parameter.

Remark 2. From Eqs. (4) and (5), the time-delayed control is designed in the competition sub-network which could prevent the consensus of the coupled agents. Fig. 2 shows that the time-delayed control in the competition sub-network when agent j is a competition neighbor of agent i, which is typically simple to be implemented. Moreover, the two kinds of time-delayed control schemes in Eqs. (4) and (5) can be introduced in a unified way as follows:

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) - (L_1 \otimes I_n)x(t) + (L_2 \otimes I_n)\int_{t-\tau}^{t} K(t-s)x(s)ds , \quad (6)$$

where $K(s) \in R$ is the kernel function. If $K(s) = e^{-\alpha s}\delta(s - \tau)$ with $\delta(s)$ being the Dirac function, Eq. (6) is consistent with Eq. (4), while if $K(s) = e^{-\alpha s}$, Eq. (6) is consistent with Eq. (5). Meanwhile, in many physical scenarios, the communication among the agents in the system displays such temporal decay [46], which is also observed in many man-made systems. Hence, the decay term of time delay τ, i.e., $e^{-\alpha\tau}$, is also considered in the time-delayed control schemes in Eqs. (4) and (5), and α is a parameter which depends on the specific situation to be modeled. Furthermore, in Eq. (4), the term $e^{-\alpha\tau}x(t-\tau)$ indicates that the increment of the multi-agent system's state at t is based on its present state and former state at $t - \tau$, while the term $\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds$ in Eq. (5) reflects the continuous influence of the past states on the current state, which indicates that the value of current state is mostly influenced by those more recent states.

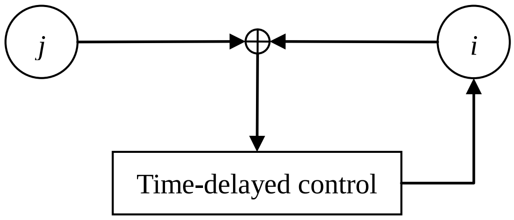

Fig. 1. the time-delayed control in the competition sub-network.

Remark 3. It should be noted that the systems described by Eqs. (4) and (5) are time-delay systems. In order to guarantee the existence of solutions in these systems, the initial conditions of Eqs. (4) and (5) should be provided, which can be the arbitrary differentiable initial functions defined in $[-\tau, 0]$. Since Eq. (3) has the initial condition $x(0)$, here the initial conditions of Eqs. (4) and (5) are set to be $x(t) = x(0)$ for $t \in [-\tau, 0]$, which can be implemented by zero-order hold.

III. MAIN RESULTS

In this section, the consensus problem of nonlinear multi-agent systems described by Eqs. (4) and (5) is investigated. The synchronization manifold method is successfully applied to the analysis of consensus in cooperation-competition networks, and some algebraic criterions are established.

Firstly, the following assumptions are needed for the derivation of the main results in this paper.

Assumption 1. For the nonlinear function $f(t,x)$, there exists constant $l_1 > 0$, such that

$$\|f(t,x) - f(t,\overline{x})\| \le l_1 \|x - \overline{x}\|, \forall t \in R,\ x, \overline{x} \in R^n . \tag{7}$$

Assumption 2. For the nonlinear function $f(t,x)$, there exists constant $l_2 > 0$, such that

$$\left[f(t,x) - f(t,\overline{x})\right]^T \left[x - \overline{x}\right] \le l_2 \left[x - \overline{x}\right]^T \left[x - \overline{x}\right], \quad \forall t \in R,\ x, \overline{x} \in R^n . \tag{8}$$

Note that **Assumption 1** is a Lipschitz condition, which is satisfied by, for example, all piecewise linear functions [43, 45]. Meanwhile, according to **Assumption 2**, it is easy to see that the inequality (8) holds when the nonlinear function $f(t,x)$ satisfies the Lipschitz condition. In fact, for any constant $\xi > 0$, one has

$$\begin{aligned}\left[f(t,x) - f(t,\overline{x})\right]^T \left[x - \overline{x}\right] &\le \frac{1}{2}\left\{\frac{\left[f(t,x) - f(t,\overline{x})\right]^T \left[f(t,x) - f(t,\overline{x})\right]}{\xi} + \xi\left[x - \overline{x}\right]^T \left[x - \overline{x}\right]\right\} \\ &\le \frac{1}{2}\left(\frac{l_1^2}{\xi} + \xi\right)\left[x - \overline{x}\right]^T \left[x - \overline{x}\right].\end{aligned}$$

But the converse situation is not true, which has been shown in [47]. Hence, for the nonlinear function $f(t,x)$, the inequality (8) is less conservative than the Lipschitz condition (7) which is usually assumed in the literature.

Then, the main result of the paper is given by the following theorem:

Theorem 1. Given a system described by Eq. (4), suppose that **Assumption 1** is satisfied, if there exist matrices $M \in M_2^N$, and $\Lambda = diag\left(\Lambda_1, \cdots, \Lambda_p\right) \in R^{p \times p}$, where $\Lambda_j > 0$ for all $j = 1, \cdots, p$, such that

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned}\Omega_{11} = & -\frac{\hat{L}_1 + \hat{L}_1^T}{2} \otimes I_n + \tau \hat{L}_1^T \hat{L}_1 \otimes I_n - \frac{1}{\tau} I_{pn} - \frac{I_p \otimes \left(C + C^T\right)}{2} \\ & + \tau\left(I_p \otimes C^T C\right) + \tau\left[\left(\hat{L}_1^T \otimes C\right) + \left(\hat{L}_1 \otimes C^T\right)\right] + \tau l_1^2 \Lambda \otimes I_n,\end{aligned}$$

$$\Omega_{12} = \frac{e^{-\alpha\tau}}{2}\left(\hat{L}_2 \otimes I_n\right) - \tau e^{-\alpha\tau} \hat{L}_1^T \hat{L}_2 \otimes I_n + \frac{1}{\tau} I_{pn} - \tau e^{-\alpha\tau}\left(\hat{L}_2 \otimes C^T\right),$$

$$\Omega_{13} = \frac{I_{pn}}{2} - \tau(I_p \otimes C^T) - \tau\left(\hat{L}_1^T \otimes I_n\right),$$

$$\Omega_{22} = \tau e^{-2\alpha\tau}\left(\hat{L}_2^T \hat{L}_2 \otimes I_n\right) - \frac{1}{\tau} I_{pn},$$

$$\Omega_{23} = \tau e^{-\alpha\tau}\left(\hat{L}_2^T \otimes I_n\right),$$

$$\Omega_{33} = \tau\left(I_{pn} - \Lambda \otimes I_n\right),$$

and $\hat{L}_1$, $\hat{L}_2$ are defined in **Lemma 1**, i.e., $ML_1 = \hat{L}_1 M$, $ML_2 = \hat{L}_2 M$, then the consensus of the system (4) can be achieved, i.e., $\lim_{t \to \infty} \left\| x_i(t) - x_j(t) \right\| = 0$, $\forall i, j = 1, \cdots, N$.

Proof: Consider the following auxiliary function

$$V(t) = \frac{1}{2} x^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right) x(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\dot{x}(s) ds d\theta, \tag{10}$$

where $M \in M_2^N$ is a $P \times N$ matrix and $\tau > 0$ is the control parameter. Then, in order to study the behavior of Eq. (4), one can calculate the increment of $V(t)$ with time, i.e.,

$$\begin{aligned}\dot{V}(t) = & -x^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\left(I_N \otimes C\right) x(t) + \tau \cdot \dot{x}^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\dot{x}(t) \\ & - x^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\left(L_1 \otimes I_n\right) x(t) + x^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\left(I_N \otimes C\right) F(t, x(t)) \\ & - \int_{t-\tau}^{t} \dot{x}^T(s)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\dot{x}(s) ds + e^{-\alpha\tau} x^T(t)\left(M \otimes I_n\right)^T\left(M \otimes I_n\right)\left(L_2 \otimes I_n\right) x(t-\tau)\end{aligned}. \tag{11}$$

According to **Lemma 1** and the property of the Kronecker product, it can be shown that $\left(M \otimes I_n\right)\left(I_N \otimes C\right) = \left(I_p \otimes C\right)\left(M \otimes I_n\right)$ and $\left(M \otimes I_n\right)\left(L_k \otimes I_n\right) = \left(\hat{L}_k \otimes I_n\right)$, $\forall k = 1, 2$, where $\hat{L}_k = ML_k G_M$. Hence, one has

$$\dot{V}(t) = -x^T(t)\left(M\otimes I_n\right)^T\left(I_p\otimes C\right)\left(M\otimes I_n\right)x(t) + \tau\cdot\dot{x}^T(t)\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\dot{x}(t)$$
$$-x^T(t)\left(M\otimes I_n\right)^T\left(\hat{L}_1\otimes I_n\right)\left(M\otimes I_n\right)x(t) + x^T(t)\left(M\otimes I_n\right)^T\left(I_p\otimes C\right)\left(M\otimes I_n\right)F(t,x(t)) \quad . \quad (12)$$
$$-\int_{t-\tau}^{t}\dot{x}^T(s)\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\dot{x}(s)ds + e^{-\alpha\tau}x^T(t)\left(M\otimes I_n\right)^T\left(\hat{L}_2\otimes I_n\right)\left(M\otimes I_n\right)x(t-\tau)$$

Next, consider the term $\int_{t-\tau}^{t}\dot{x}^T(s)\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\dot{x}(s)ds$, then it follows from **Lemma 2** that

$$\int_{t-\tau}^{t}\dot{x}^T(s)\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\dot{x}(s)ds \ge \frac{1}{\tau}\left[x(t)-x(t-\tau\right]^T\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\left[x(t)-x(t-\tau\right]. \quad (13)$$

Meanwhile, consider the term $F^T(t,x(t))\left(M\otimes I_n\right)^T\Lambda\left(M\otimes I_n\right)F(t,x(t))$, since $F(t,x)=\left[f^T(t,x_1),\cdots,f^T(t,x_N)\right]^T$, one has

$$(M\otimes I_n)\cdot F(t,x) = \begin{bmatrix} f(t,x_{j_{11}}) - f(t,x_{j_{12}}) \\ \vdots \\ f(t,x_{j_{i1}}) - f(t,x_{j_{i2}}) \\ \vdots \\ f(t,x_{j_{P1}}) - f(t,x_{j_{P2}}) \end{bmatrix}, \quad (14)$$

where j_{i1} and j_{i2} denote the column indexes of the first and second nonzero elements, respectively, in the i th row. Then, according to **Assumption 1**, it can be shown that

$$\begin{aligned} &F^T(t,x(t))\left(M\otimes I_n\right)^T\left(\Lambda\otimes I_n\right)\left(M\otimes I_n\right)F(t,x(t)) \\ &= \sum_{i=1}^{P}\Lambda_i\left\|f(t,x_{j_{i1}}) - f(t,x_{j_{i2}})\right\|^2 \\ &\le \sum_{i=1}^{P}\Lambda_i l_1^2\left\|x_{j_{i1}} - x_{j_{i2}}\right\|^2 \\ &= l_1^2 x^T(t)\left(M\otimes I_n\right)^T\left(\Lambda\otimes I_n\right)\left(M\otimes I_n\right)x(t) \end{aligned} \quad . \quad (15)$$

From Eqs. (13)-(15), Eq. (12) can be rewritten by

$$\dot{V}(t) \le -x^T(t)\left(M\otimes I_n\right)^T\left(I_p\otimes C\right)\left(M\otimes I_n\right)x(t) - x^T(t)\left(M\otimes I_n\right)^T\left(\hat{L}_1\otimes I_n\right)\left(M\otimes I_n\right)x(t)$$
$$+x^T(t)\left(M\otimes I_n\right)^T\left(I_p\otimes C\right)\left(M\otimes I_n\right)F(t,x(t)) + \tau\cdot\dot{x}^T(t)\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\dot{x}(t)$$
$$+e^{-\alpha\tau}x^T(t)\left(M\otimes I_n\right)^T\left(\hat{L}_2\otimes I_n\right)\left(M\otimes I_n\right)x(t-\tau) + l_1^2 x^T(t)\left(M\otimes I_n\right)^T\left(\Lambda\otimes I_n\right)\left(M\otimes I_n\right)x(t). \quad (16)$$
$$-F^T(t,x(t))\left(M\otimes I_n\right)^T\left(\Lambda\otimes I_n\right)\left(M\otimes I_n\right)F(t,x(t))$$
$$-\frac{1}{\tau}\left[x(t)-x(t-\tau)\right]^T\left(M\otimes I_n\right)^T\left(M\otimes I_n\right)\left[x(t)-x(t-\tau)\right]$$

Now taking Eq. (4) into Eq. (16), one obtains

$$
\begin{aligned}
\dot{V}(t) \le & -x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) + e^{-\alpha\tau}x^T(t)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)x(t-\tau) \\
& + \tau\cdot x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) - \frac{1}{\tau}x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t) \\
& - 2\tau e^{-\alpha\tau}x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)x(t-\tau) \\
& + \tau e^{-\alpha\tau}x^T(t-\tau)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)x(t-\tau) \\
& + \frac{2}{\tau}x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t-\tau) - \frac{1}{\tau}x^T(t-\tau)(M\otimes I_n)^T(M\otimes I_n)x(t-\tau) \\
& - x^T(t)(M\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) + x^T(t)(M\otimes I_n)^T(M\otimes I_n)F(t,x(t)) \\
& + \tau\cdot x^T(t)(M\otimes I_n)^T(I_p\otimes C)^T(I_p\otimes C)(M\otimes I_n)x(t) \\
& + \tau\cdot x^T(t)(M\otimes I_n)^T(I_p\otimes C)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) \\
& - \tau e^{-\alpha\tau}x^T(t)(M\otimes I_n)^T(I_p\otimes C)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)x(t-\tau) \\
& + \tau\cdot x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) \\
& - \tau e^{-\alpha\tau}x^T(t-\tau)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) \\
& - \tau\cdot F^T(t,x(t))(M\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) \\
& + \tau\cdot F^T(t,x(t))(M\otimes I_n)^T(M\otimes I_n)F(t,x(t)) \\
& - \tau\cdot F^T(t,x(t))(M\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) \\
& + \tau e^{-\alpha\tau}F^T(t,x(t))(M\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)x(t-\tau) \\
& - \tau\cdot x^T(t)(M\otimes I_n)^T(I_p\otimes C)^T(M\otimes I_n)F(t,x(t)) \\
& - \tau\cdot x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)^T(M\otimes I_n)F(t,x(t)) \\
& + \tau e^{-\alpha\tau}x^T(t-\tau)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)^T(M\otimes I_n)F(t,x(t)) \\
& + l_1^2x^T(t)(M\otimes I_n)^T(\Lambda\otimes I_n)(M\otimes I_n)x(t) \\
& - F^T(t,x(t))(M\otimes I_n)^T(\Lambda\otimes I_n)(M\otimes I_n)F(t,x(t))
\end{aligned}
\quad . \ (17)
$$

Let $\eta = \left[x^T(t)(M\otimes I_n)^T, x^T(t-\tau)(M\otimes I_n)^T, F(t,x(t))(M\otimes I_n)^T\right]^T$. From Eq. (17), one has $\dot{V}(t) \le \eta^T\Omega\eta$, and $\dot{V}(t)\le 0$ under the condition (9). Hence, as a consequence of Lyapunov Theory [29], $\|(M\otimes I_n)x(t)\| \to 0$ as $t\to\infty$. From the definition of M_2^N, it can be concluded that $\lim_{t\to\infty}\|x_i(t)-x_j(t)\| = 0$, $\forall i, j = 1,\cdots,N$. Thus the consensus of the system (4) can be achieved.

Remark 4. In **Theorem 1**, according to the synchronization manifold, the consensus problem of multi-agent systems with inherent nonlinear dynamics is solved asymptotically. Moreover, it should be noted that the multi-agent systems described by Eq. (4) has a more general network topology than those considered in the literature [4, 6, 10, 11, 12, 19, 21, 32]. Different from these studies, where only the cooperation networks with the nonnegative weight matrices, in this paper, we consider cooperation-competition networks. For this kind of networks, the entries of the weight matrices may be negative, which can lead to the negative off-diagonal entries of the Laplacian matrices, and in this situation, we say the coupled agents compete with each other.

Generally, it is necessary to have a directed spanning tree in order to achieve consensus on cooperative networks without any extra controller. However, in **Theorem 1**, a directed spanning tree in cooperation sub-network is not a necessary condition of consensus any longer, when considering the extra competition relationships between agents and the time-delayed control. In fact, this is mainly because the competition relationships may increase the chance of information exchange between the agents, while the extra time-delayed control guarantees that the errors between the coupled competitive agents converge to zero asymptotically. In other words, according to the proof of **Theorem 1**, it is easy to see that the competition relationships could help the agents achieve consensus under the time-delayed control designed in the competition sub-network.

Remark 5. According to **Theorem 1**, it can be seen that the consensus of Eq. (4) cannot be reached for a sufficiently small τ or sufficiently large α and τ. In fact, if $\tau = 0$, then Eq. (4) does not have the time-delayed control, and due to the competition relationship in the network topology, Eq. (4) could not achieve consensus spontaneously. Moreover, if $\tau = \infty$ or $\alpha = \infty$, then Eq. (4) can be rewritten as follows:

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) - (L_1 \otimes I_n)x(t), \tag{18}$$

which degenerates to the pure cooperation case with no competition relationships between any pair of agents. Since it is not assumed that the cooperation sub-network is strongly connected or contains a directed spanning tree in **Theorem 1**, Eq. (18) could not achieve consensus either.

Next, the consensus problem in a pure competition network is considered, where all the coupled agents compete with each other and thus the entries of the weight matrix are negative or zeros, i.e., $L_1 = 0$. Eq. (4) has the form:

$$\dot{x}(t) = -(I_N \otimes C)x(t) + F(t, x(t)) + e^{-\alpha\tau}(L_2 \otimes I_n)x(t-\tau). \tag{19}$$

Then, according to **Lemma 1**, it can be obtained that $\hat{L}_1 = 0$. The following corollary gives a sufficient condition to establish consensus in such a competition network.

Corollary 1. Given a system described by Eq. (19), suppose that **Assumption 1** is satisfied, if there exist matrices $M \in M_2^N$, and $\Lambda = diag\left(\Lambda_1, \cdots, \Lambda_p\right) \in R^{p\times p}$, where $\Lambda_j > 0$ for all $j = 1, \cdots, p$, such that

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} \\ * & \tilde{\Omega}_{22} & \tilde{\Omega}_{23} \\ * & * & \tilde{\Omega}_{33} \end{bmatrix} < 0, \tag{20}$$

where

$$\tilde{\Omega}_{11} = -\frac{1}{\tau} I_{pn} - \frac{I_p \otimes \left(C + C^T\right)}{2} + \tau\left(I_p \otimes C^T C\right) + \tau l_1^2 \Lambda \otimes I_n,$$

$$\tilde{\Omega}_{12} = \frac{e^{-\alpha\tau}}{2}\left(\hat{L}_2 \otimes I_n\right) + \frac{1}{\tau} I_{pn} - \tau e^{-\alpha\tau}\left(\hat{L}_2 \otimes C^T\right),$$

$$\tilde{\Omega}_{13} = \frac{I_{pn}}{2} - \tau(I_p \otimes C^T),$$

$$\tilde{\Omega}_{22} = \tau e^{-2\alpha\tau}\left(\hat{L}_2^T \hat{L}_2 \otimes I_n\right) - \frac{1}{\tau} I_{pn},$$

$$\tilde{\Omega}_{23} = \tau e^{-\alpha\tau}\left(\hat{L}_2^T \otimes I_n\right),$$

$$\tilde{\Omega}_{33} = \tau\left(I_{pn} - \Lambda \otimes I_n\right),$$

and $\hat{L}_2$ is defined in **Lemma 1**, i.e., $ML_2 = \hat{L}_2 M$, then the consensus of the system (19) can be achieved, i.e., $\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = 0$, $\forall i, j = 1, \cdots, N$.

Note that, the time-delayed control scheme in **Theorem 1** and **Corollary 1** indicates that the increment of the multi-agent system's state at t is based on its present state and former state at $t-\tau$. In the following theorem, the consensus problem of Eq. (5) is considered, where the increment of the state at t is affected by the former states on the interval $[t-\tau, t]$, i.e., the term $\int_{t-\tau}^{t} e^{-\alpha(t-s)} x(s) ds$ which indicates that the increment of the current state is mostly influenced by those more recent states.

Theorem 2. Given a system described by Eq. (5), suppose that **Assumption 2** is satisfied, if there exists a matrix $M \in M_2^N$, such that

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & \Delta_{22} \end{bmatrix} < 0, \tag{21}$$

where

$$\Delta_{11} = -\frac{I_p \otimes C + I_p \otimes C^T}{2} - \frac{\hat{L}_1 \otimes I_n + \hat{L}_1^T \otimes I_n}{2} + \left(\tau + l_2\right) I_{pn},$$

$$\Delta_{12} = \frac{\hat{L}_2 \otimes I_n}{2}, \quad \Delta_{22} = -\frac{1}{\tau} I_{pn},$$

and $\hat{L}_1$, $\hat{L}_2$ are defined in **Lemma 1**, i.e., $ML_1 = \hat{L}_1 M$, $ML_2 = \hat{L}_2 M$, then the consensus of the system (5) can be achieved, i.e., $\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = 0$, $\forall i, j = 1, \cdots, N$.

Proof: Consider the following Lyapunov function

$$V(t) = \frac{1}{2} x^T(t)\left(M \otimes I_n\right)^T \left(M \otimes I_n\right) x(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t} e^{-2\alpha(t-s)} x^T(s)\left(M \otimes I_n\right)^T \left(M \otimes I_n\right) x(s) ds d\theta, \tag{22}$$

where $M \in M_2^N$ is a $P \times N$ matrix, and $\tau > 0$ is the control parameter. Then, according to Eq. (5), it is calculated by

$$\begin{aligned}\dot{V}(t) &= x^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{x}(t) + \left[\int_{-\tau}^{0}\int_{t+\theta}^{t} e^{-2\alpha(t-s)}x^T(s)(M\otimes I_n)^T(M\otimes I_n)x(s)dsd\theta\right]' \\ &= -x^T(t)(M\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) - x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) \\ &\quad + x^T(t)(M\otimes I_n)^T(M\otimes I_n)F(t,x(t)) + x^T(t)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds \\ &\quad + \int_{-\tau}^{0} x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t)d\theta - \int_{-\tau}^{0} e^{2\alpha\theta}x^T(t+\theta)(M\otimes I_n)^T(M\otimes I_n)x(t+\theta)d\theta \\ &\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t}(-2\alpha)\cdot e^{-2\alpha(t-s)}x^T(s)(M\otimes I_n)^T(M\otimes I_n)x(s)dsd\theta\end{aligned} \quad (23)$$

Next, consider the term $x^T(t)(M\otimes I_n)^T(M\otimes I_n)F(t,x(t))$, according to **Assumption 2**, one obtains

$$\begin{aligned}x^T(t)(M\otimes I_n)^T(M\otimes I_n)F(t,x(t)) &= \sum_{i=1}^{p}\left[f(t,x_{j_{i1}}) - f(t,x_{j_{i2}})\right]^T\left(x_{j_{i1}} - x_{j_{i2}}\right) \\ &\le l_2\sum_{i=1}^{p}\left(x_{j_{i1}} - x_{j_{i2}}\right)^T\left(x_{j_{i1}} - x_{j_{i2}}\right) \\ &= l_2 x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t)\end{aligned} , \quad (24)$$

where j_{i1} and j_{i2} denote the column indexes of the first and second nonzero elements, respectively, in the i th row. Hence, Eq. (23) can be rewritten by

$$\begin{aligned}\dot{V}(t) &\le -x^T(t)(M\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) - x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) \\ &\quad + l_2x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t) + x^T(t)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds \\ &\quad + \tau\cdot x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t) - \int_{t-\tau}^{t} e^{-2\alpha(t-s)}x^T(s)(M\otimes I_n)^T(M\otimes I_n)x(s)ds \\ &\le -x^T(t)(M\otimes I_n)^T(I_p\otimes C)(M\otimes I_n)x(t) - x^T(t)(M\otimes I_n)^T(\hat{L}_1\otimes I_n)(M\otimes I_n)x(t) \\ &\quad + l_2x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t) + x^T(t)(M\otimes I_n)^T(\hat{L}_2\otimes I_n)(M\otimes I_n)\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds \\ &\quad + \tau\cdot x^T(t)(M\otimes I_n)^T(M\otimes I_n)x(t) \\ &\quad - \frac{1}{\tau}\left[\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds\right]^T(M\otimes I_n)^T(M\otimes I_n)\left[\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds\right]\end{aligned} , \quad (25)$$

Let $\varsigma = \left[x^T(t)(M\otimes I_n)^T, \left(\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds\right)^T(M\otimes I_n)^T\right]^T$, one has $\dot{V}(t) \le \varsigma^T\Delta\varsigma < 0$, $\forall\varsigma \ne 0$. By similar analysis in **Theorem 1**, it can be concluded that $\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = 0$, $\forall i,j = 1,\cdots,N$. Thus the consensus of the system (5) can be achieved.

Remark 6. In **Theorem 2**, the matrix Δ is independent of the parameter α, which is different from **Theorem 1**. However, it is found that the consensus rate of Eq. (5) can be influenced by the parameter α through a number of simulations. Therefore, it is possible to increase convergence speed by appropriately choosing the value of α, which will be carefully discussed in the next section.

When a pure competition network is considered, Eq. (5) has the form:

$$\dot{x}(t) = -(I_N\otimes C)x(t) + F(t,x(t)) + (L_2\otimes I_n)\int_{t-\tau}^{t} e^{-\alpha(t-s)}x(s)ds , \quad (26)$$

and the following corollary gives a sufficient condition to establish consensus of the multi-agent systems described by Eq. (26).

Corollary 2. Given a system described by Eq. (26), suppose that **Assumption 2** is satisfied, if there exists a matrix $M \in M_2^N$, such that

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & \Delta_{22} \end{bmatrix} < 0, \tag{27}$$

where

$$\Delta_{11} = -\frac{I_p \otimes C + I_p \otimes C^T}{2} + (\tau + l_2) I_{pn},$$

$$\Delta_{12} = \frac{\hat{L}_2 \otimes I_n}{2}, \quad \Delta_{22} = -\frac{1}{\tau} I_{pn},$$

and $\hat{L}_2$ is defined in **Lemma 1**, i.e., $ML_2 = \hat{L}_2 M$, then the consensus of the system (26) can be achieved, i.e., $\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0$, $\forall i, j = 1, \cdots, N$.

According to the condition (27), the algebraic criteria about the consensus problem of Eq. (26) is provided as follows:

Corollary 3. Given a system described by Eq. (26), suppose that **Assumption 2** is satisfied, if the control parameter $\tau < \min\left\{ \frac{\lambda_{\min}\left(C + C^T\right)}{2} - l_2 - \frac{1}{2}, \frac{2}{\lambda_{\max}\left(\hat{L}_2^T \hat{L}_2\right)} \right\}$, and $\hat{L}_2$ is defined in **Lemma 1**, then the consensus of the system (26) can be achieved, i.e., $\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0$, $\forall i, j = 1, \cdots, N$.

IV. NUMERICAL SIMULATION

In this section, two numerical examples are provided in order to validate the proposed theoretical results.

In the simulation, the agents move in a 3-dimensional space, and the nonlinear intrinsic dynamics are governed by the following Chua's oscillator in [48]:

$$f(t, x_i(t)) = \begin{bmatrix} \theta\left(x_{i2} - x_{i1} - h(x_{i1})\right) \\ x_{i1} - x_{i2} + x_{i3} \\ -\sigma \sin(w x_{i1}) - \gamma x_{i2} - \beta x_{i3} \end{bmatrix}, \tag{28}$$

where $x_i = \left(x_{i1}, x_{i2}, x_{i3}\right)^T \in R^3$, and $h(x_{i1}) = b x_{i1} + \frac{b-a}{2}\left[\left|x_{i1} + 1\right| - \left|x_{i1} - 1\right|\right]$. Moreover, the parameters in Eq. (28) are chosen as $\theta = 1$, $\sigma = 6.8$, $\beta = 0.5$, $\gamma = 1.5$, $w = 0.5$, $a = -1.2$, $b = -0.8$, and the matrix C in Eq. (1) is as follows:

$$C = \begin{bmatrix} 1.5416 & -0.012 & -0.068 \\ -0.012 & 1.6312 & 0.023 \\ -0.068 & 0.023 & 1.4359 \end{bmatrix}. \tag{39}$$

Next, we consider ten agents with the interaction topology given in Fig. 2, which does not contain a directed spanning tree. Here, the directed solid lines represent the competition relationship between agents, such as agent 7 and agent 1. In particular, agent 7 wants to have the states that deviate from those of agent 1, and thus may prevent the consensus of the coupled agents, while, the directed dotted lines represent the cooperation relationship between agents, such as agent 7 and agent 3. On the contrary, agent 3 hopes to have close states with agent 7, and thus benefits the consensus of the coupled agents.

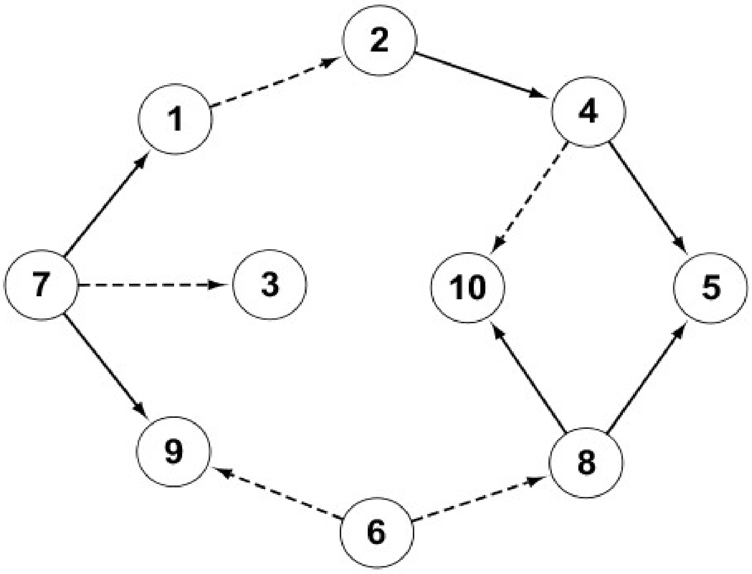

Fig. 2. The interaction topology of ten agents.

Moreover, the weight matrix A is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix}, \tag{30}$$

As we can see, here, the interaction topology of these ten agents does not have a directed spanning tree. Meanwhile, the multi-agent system includes both cooperation and competition among agents denoted by the elements 1 and -2 in A, respectively. Initially, the agents are randomly distributed in a three-dimensional cube space $[12, 21]\times[12, 21]\times[12, 21]$. Moreover, the total error of the multi-agent system with respect to agent 1 is defined by

$$e(t) \triangleq \frac{1}{10}\sum_{j=1}^{10}\sqrt{\sum_{i=1}^{3}\left[x_{1i}(t)-x_{ji}(t)\right]^2}, \tag{31}$$

When there is no time-delayed control in the competition sub-network, the evolutions of

$x(t)$ and $e(t)$ of the ten agents are shown in Figs. 3 and 4, respectively. In this case, the agents in the system described by Eq. (3) cannot achieve consensus.

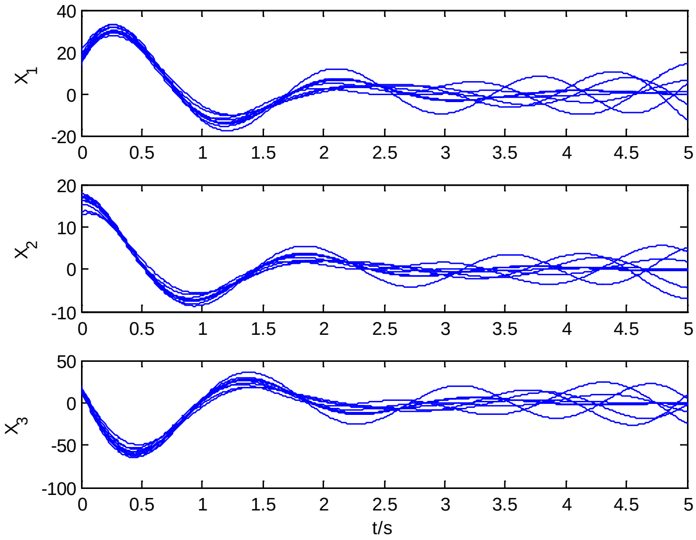

Fig. 3. The evolution of $x(t)$ without control.

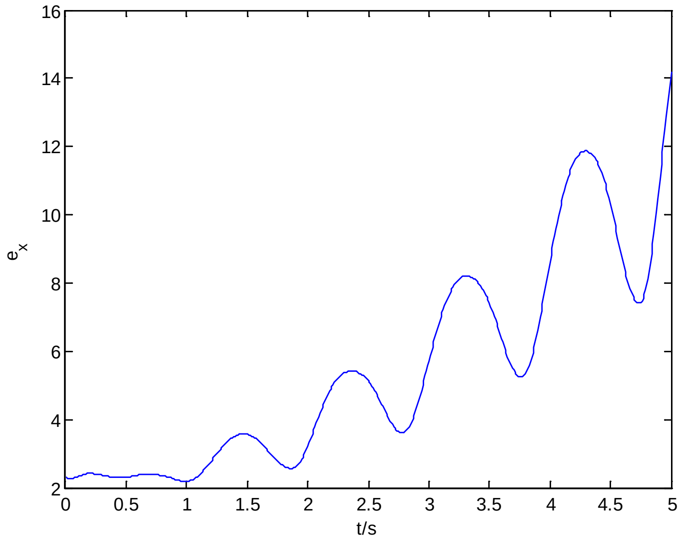

Fig. 4. The evolution of $e(t)$ without control.

Next, two kinds of time-delayed control schemes are designed in the competition sub-network in order to achieve consensus, based on the analytical results. Here, the matrices M and G_M in **Lemma 1** are set to be

$$M=\begin{bmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & & \ddots & \\ & & & 1 & -1\end{bmatrix}\in R^{9\times 10},\quad G_M=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & 1 & 1 & \cdots & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\in R^{10\times 9}. \qquad (32)$$

Example 1. Consider the multi-agent system (4) with the time-delayed control in the competition sub-network, and the parameters in the time-delayed control are chosen as $\alpha = 0.5$ and $\tau = 0.75$. Based on **Theorem 1**, a feasible solution of Λ can be obtained by $\Lambda = 1.2I_9$ and $\Omega < 0$, which means that the agents of the system can achieve consensus. Here, the evolutions of $x(t)$ and $e(t)$ of the agents with the time-delayed control are shown in Figs. 5 and 6, respectively.

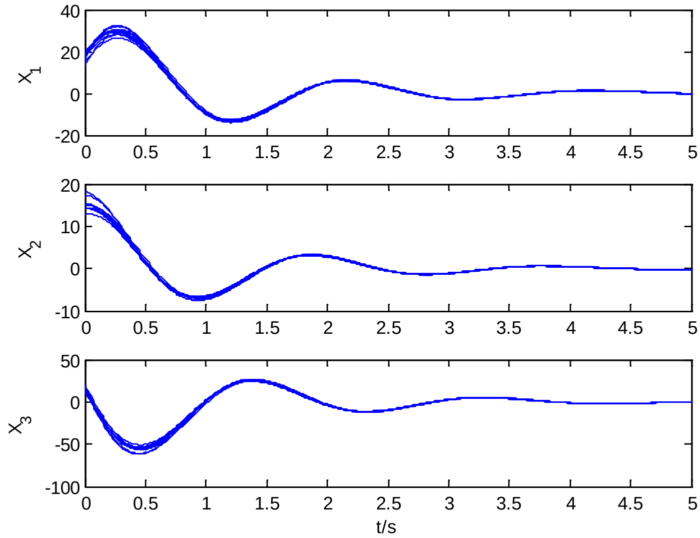

Fig. 5. The evolution of $x(t)$ with the time-delayed control (4).

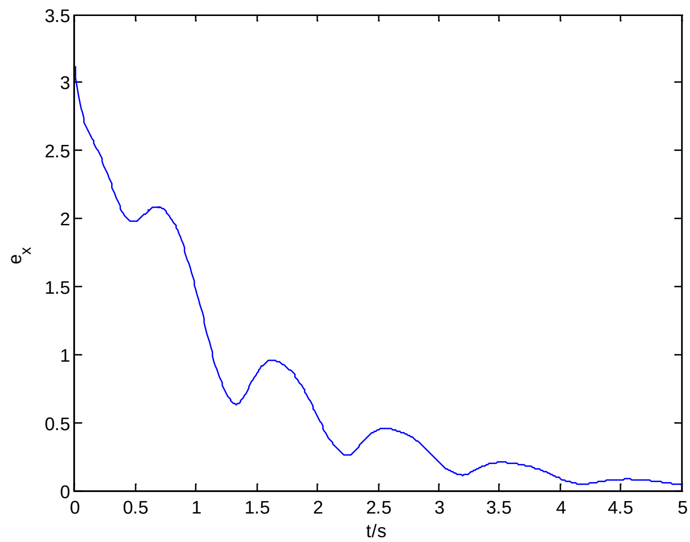

Fig. 6. The evolution of $e(t)$ with the time-delayed control (4).

Example 2. Consider the multi-agent system (5) with the time-delayed control in the competition sub-network. Based on **Theorem 2**, the feasible parameters in the time-delayed control are chosen as $\alpha = 1.5$ and $\tau = 0.55$, and thus we have $\Delta < 0$, which also means that now the agents of the system can achieve consensus. Here, the evolutions of $x(t)$ and $e(t)$ of the agents with the time-delayed control are shown in Figs. 7 and 8, respectively.

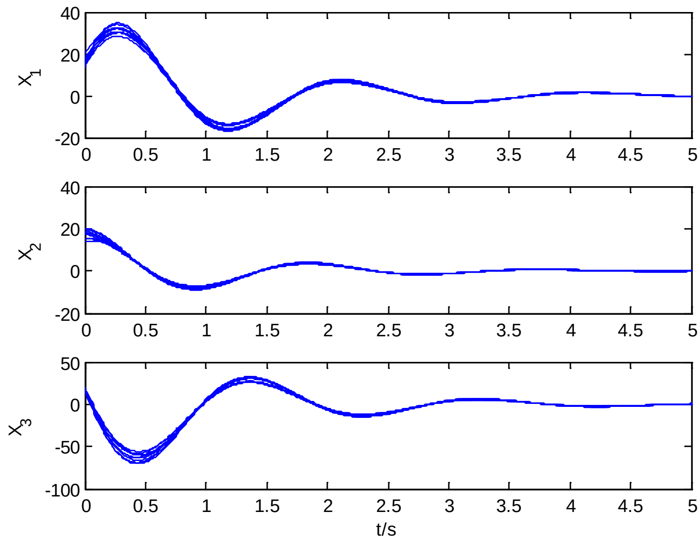

Fig. 7. The evolution of $x(t)$ with the time-delayed control (5).

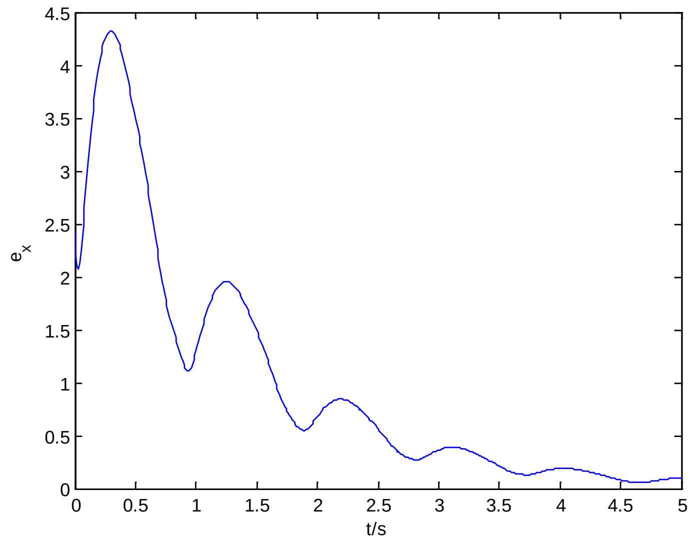

Fig. 8. The evolution of $e(t)$ with the time-delayed control (5).

According to **Theorem 2**, the matrix Δ is independent of the parameter α, however, this parameter can indeed slightly influence the consensus rate of the multi-agent system (5). To gain the further insight in the influence, we do more numerical simulations on the multi-agent system shown in Fig. 2. Here, suppose that the settling time of the multi-agent system (5) is regarded as the time when the corresponding error function $e(t)$ reaches 0.01, then the relationship between the settling time $e(t)$ and the parameter α is shown in Fig. 9. Moreover, it can be seen that when $\alpha \leq 5$, the multi-agent system with appropriately small α has excellent performance, or say, has a short settling time, and when $\alpha > 5$, on the contrary, the multi-agent system with large α has a short settling time.

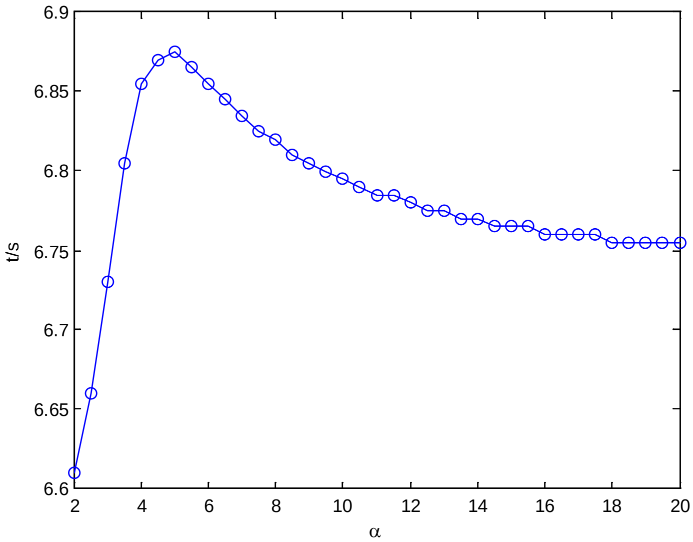

Fig. 9. The settling time of $e(t)$ with different α .

VI. CONCLUSION

This paper discussed the consensus problem of first-order agents in the cooperation-competition network. In this study, the whole network was divided into two sub-networks, i.e., the cooperation sub-network and the competition sub-network, and then two kinds of time-delayed control schemes have been designed in the competition sub-network. Based on the graph theory, the Lyapunov technique, and the synchronization manifold method, it is proved that, without assuming that the network is strongly connected or contains a directed spanning tree, the agents in the cooperation-competition network with the time-delayed control can reach the consensus asymptotically. Besides, as an extension, the consensus problem in the pure competition network has also been studied. Numerical simulations have validated our theoretical results. Note that, here, all the agents are governed by the same nonlinear intrinsic dynamics, and our future work will focus on the situations where the agents do not share the same intrinsic dynamics.

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