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Second-order consensus of multi-agent systems in the cooperation–competition network with switching topologies: A time-delayed impulsive control approach

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ABSTRACT

This paper investigates the consensus problem of second-order multi-agent systems with switching topologies by designing a time-delayed impulsive consensus control scheme. All the agents are governed by the same nonlinear intrinsic dynamics. In this study, agents can cooperate or compete with each other, i.e., the elements in the weight matrix of the coupling graph can be either positive or negative. By establishing a comparison system, a new comparison principle method is successfully applied to study such consensus problem. Then, several effective sufficient conditions are attained without assuming that the interaction topology is strongly connected or contains a directed spanning tree. Meanwhile, the exponential consensus rate is also obtained. Finally, simulation results are presented to validate the effectiveness of the theoretical analysis.

1. Introduction

A multi-agent system is always composed of many interconnected agents, in which agents represent individual elements with their own dynamics and edges represent the relationships between their dynamics. Multi-agent systems are ubiquitous in the real world, such as electrical power grids, food webs, global economic markets, social networks and so on [1–4].

The consensus problem of multi-agent systems has attracted much attention from a variety of areas, since the research on such problem not only helps better understand the mechanisms of natural collective phenomena, such as avoiding predators and increasing the chance of foraging food, but also provides useful ideas to develop formation control and distributed cooperative control for coordination of multiple mobile autonomous robots.

Generally, the agents in these systems usually have second-order dynamics [5–7]. At the same time, the edges between the agents are always variable dynamically. That is, since agents move around in space, some of the existing edges may fail to be sustained due to the appearance of obstacles between the communicating agents, meanwhile, new edges may be also newly created, when two agents enter into the sensing ranges of each other. Therefore, it seems more practical to study consensus problem of second-order agent dynamics with switching topologies [8]–[9]. Ren [10] analyzed consensus algorithms for second-order dynamics in four cases, where the conditions for second-order consensus were derived according to the interaction topology. Xie and Wang [11] investigated the consensus problem of second-order agents with switching topologies, and a linear consensus protocol was established for solving such a consensus problem. Yu et al. [12] considered the second-order consensus in multi-agent dynamical systems with sampled position data, and a necessary and sufficient condition for reaching consensus of the system in this setting was established.

Over decades, numerous interesting results of consensus have been obtained by considering different subsistent limitations in multi-agent systems, such as time delay [13–15], time varying coupling [16–19], higher-order consensus [20–22]. Various control techniques such as adaptive control, pinning control, and impulsive control, have been proposed to achieve consensus of multi-agent systems. Compared with other control techniques, impulsive control [23] behaves better in improving transient response, increasing the efficiency of bandwidth usage, and can provide an effective mechanism to cope with multi-agent systems with large

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uncertainties. For example, Hui [24] developed a novel impulsive control framework to address consensus problem of multi-agent systems, under which finite-time consensus can be achieved, and thus the performance of the closed-loop system can be improved. Liu and Hill [25] investigated the global consensus problem between a multi-agent system and a known goal signal by designing an impulsive consensus control scheme. In their work, the multi-agent system involved the uncertainties, nonidentical nodes, and coupling time-delays, and the consensus criteria were expressed in terms of linear matrix inequalities (LMIs) and algebraic inequalities. In practical applications, time-delay [26] is always unavoidable due to the communication congestion of the network and the finite switching and spreading speed of the hardware and circuit implementation, ignorance of which may cause the multi-agent system to diverge or oscillate. Guan et al. [27] considered the consensus problem with system delay and multiple coupling delays via impulsive distributed control, and introduced the concept of control topology to describe the whole controller structure.

With this background, we investigate a second-order consensus of multi-agent systems with switching topologies via time-delayed impulsive control. Here, all agents are governed by nonlinear intrinsic dynamics. In this control scheme, the time-delayed impulsive control signal is designed as the input of each agent at switching instances. Obviously, such control scheme is typically simple to be implemented. By establishing a comparison system, a new comparison principle method is successfully applied to the study of the consensus problem of multi-agent systems, and some consensus criteria are obtained without assuming that the interaction topology is strongly connected or contains a directed spanning tree. Furthermore, it should be noted that here the weight matrix of the coupling graph is not assumed to be nonnegative, i.e., the off-diagonal entries of the Laplacian matrix may be negative. Actually, this means that the weight may prevent, rather than benefit, the consensus of the coupled agents. In such a situation, we say the coupled agents compete with each other, which just means that they are not willing to cooperate with each other to achieve consensus. Therefore, the situation introduced here is called the cooperation–competition network, and is more complicated than the cases in most of the former works where only cooperation is considered. This hypothesis is reasonable because it has been revealed that the agents in many real-world systems can either cooperate or compete with each other. For instance, Tao et al. [28] argued that, in a business environment, selfish agents work for their own goals and compete with each other by nature, but on the other hand, it is also desirable for them to be cooperative in order to achieve efficient mutually beneficial win-win solutions. Bertness and Yeh [29] indicated that there is a balance between positive and negative interactions in the plant recruitment. Besides, Lorenceau and Zago [30] provided psychophysical and physiological evidences that long-range excitatory and inhibitory interactions within primary visual cortex modulate perceptual linking when recovering the velocity of objects moving in the visual field.

The rest of the paper is organized as follows. In Section 2, some basic definitions in graph theory and mathematical preliminary results are presented. In Section 3, a time-delayed impulsive control scheme is provided for second-order multi-agent system with switching topologies, and some consensus criteria are obtained in Section 4. In Section 5, numerical simulations are implemented to demonstrate the analytic results. Finally, the paper is concluded in Section 6.

2. Problem formulation

In this section, some basic definitions in graph theory and preliminary mathematical results are firstly introduced for subsequent use.

The mathematical notations which will be employed in the rest of the paper are presented as follows. Let R^n denote the n-dimensional real vector space. The Euclidean norms of a vector $x = (x_1, \dots, x_n)^T \in R^n$ and a matrix $A \in R^{n\times n}$ are denoted by $\|x\| \triangleq \sqrt{(\sum_{i=1}^n x_i^2)}$ and $\|A\| \triangleq \sqrt{\lambda_{\max}(A^TA)}$, respectively, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the matrix A. Moreover, the spectral radius of a matrix A is denoted by $\rho(A) \triangleq \max_i\{|\lambda_i|\}$ with λ_i being one eigenvalue of A. The identity matrix of order n is denoted by I_n, and the upper right Dini derivative of $f(t) : [0, +\infty) \to R$ is denoted by $D^+f(t) = \lim_{h\to 0^+} \sup \frac{f(t+h)-f(t)}{h}$. The Kronecker product of matrices A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix},$$

which satisfies the following properties:

$$\|I_P \otimes A\| = \|A \otimes I_N\| = \|A\|,$$
$$(A + B) \otimes C = A \otimes C + B \otimes C,$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

2.1. Topology description

In general, information exchanges between agents in a multi-agent system can be modeled by directed or undirected graphs [1,31]. Let $\Re = (V, \zeta, A)$ be a weighted directed graph of N agents with a set of nodes $V = \{\pi_1, \pi_2, \dots, \pi_N\}$, a set of edges $\zeta \subseteq V \times V$, and a weight matrix $A = [a_{ij}]$, which are used to represent the communication topology with the node indexes belonging to a finite index set $I = \{1, 2, \dots, N\}$. An edge of $\Re$ is denoted by $e_{ij} = (\pi_i, \pi_j)$ with each edge associated with a nonzero weight, i.e. $(\pi_i, \pi_j) \in \zeta \Leftrightarrow a_{ij} \neq 0$, then the neighbor set of node π_i is denoted by $N_i = \{\pi_j \,|\, (\pi_j, \pi_i) \in \zeta\}$. Note that here $a_{ij} < 0$ also makes sense, which means that agents i and j compete with each other, and thus prevent the consensus of the system.

Here we assume that $a_{ii} = 0$ for all $i \in I$, which means that self-loop is not allowed in the graph. The Laplacian matrix $L = [l_{ij}] \in R^{N\times N}$ associated with the weight matrix A is defined as

$$l_{ij} = -a_{ij}, \quad i \neq j,$$
$$l_{ii} = \sum_{j\in N_i} a_{ij}, \tag{1}$$

which ensures that $\sum_{j=1}^N l_{ij} = 0, \forall i \in I$.

2.2. Mathematical preliminaries

Let $T(\varepsilon)$ denote the set of matrices with the sum of the elements in each row equal to the real number ε. Denote by M_1^N the set of matrices with N columns and each row containing exactly two nonzero elements, i.e., 1 and -1. Then the subset $M_2^N \subseteq M_1^N$ is defined as follows: if $M = [m_{ij}] \in M_2^N$, then for each pair of column indices i and j, there exist indices $j_1, j_2, \dots, j_l$ and $p_1, p_2, \dots, p_{l-1}$ such that $m_{p_q,j_q} \neq 0$ and $m_{p_q,j_{q+1}} \neq 0$ for $q = 1, 2, \dots, l-1$, with $j_1 = i$ and $j_l = j$.

Similar to [15,32], the following lemma will be used in the derivation of the main results.

Lemma 1. *Let $M \in M_2^N$ be a $P \times N$ matrix and H be a $N \times N$ matrix. Then there exists an $N \times P$ matrix G_M such that $MH = \hat{H}M$ with $\hat{H} = MHG_M$. Furthermore, if $M \in M_2^N$ is an $(N-1) \times N$ matrix, then $MG_M = I_{N-1}$.*

3. System model

In this section, a new time-delayed impulsive control scheme is presented for second-order multi-agent systems with switching topologies. Here, the time-delayed impulsive control signal is designed as the input of each agent at switching instances.

Denote by $\wp = \{L_1, L_2, \dots, L_m\}$ the set of Laplacian matrices for all possible topologies with $\Im = \{1, 2, \cdots, m\}$ as its index set. Here a switching topology is defined by a switching signal $\sigma(t) : [0, +\infty) \to \Im$, which is represented by a piecewise constant function $\{\sigma_k\} : (t_{k-1}, t_k] \to \sigma_k$ with the time sequence $\{t_k\}$ satisfying:

$$0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \quad \lim_{k\to\infty} t_k = \infty. \tag{2}$$

Consider a nonlinear second-order multi-agent system composed of N coupled autonomous agents with switching topologies:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = f(x_i(t), v_i(t)) + \alpha \sum_{j\in N_i(t)} a_{ij}\left[(x_j - x_i) + (v_j - v_i)\right], \\ \quad t \in (t_{k-1}, t_k], i \in I, \end{cases} \tag{3}$$

where $x_i(t), v_i(t) \in R^n$, $\alpha > 0$ is the coupling strength, and $f(x_i, v_i) \in R^n$ is the intrinsic dynamic of agent i. For simplification, just like most recent works on second-order consensus problems with nonlinear agent dynamics [16,33], here it is supposed that each agent has the same intrinsic dynamics.

Denote $x = (x_1^T, x_2^T, \dots, x_N^T)^T$, $v = (v_1^T, v_2^T, \dots, v_N^T)^T$, $F(x, v) = \left[f^T(x_1, v_1), \dots, f^T(x_N, v_N)\right]^T$, then Eq. (3) can be rewritten as Eq. (4):

$$\begin{bmatrix}\dot{x}(t)\\ \dot{v}(t)\end{bmatrix} = \begin{bmatrix}0\\ F(x,v)\end{bmatrix} + \begin{bmatrix}0 & I_N\otimes I_n\\ -\alpha(L_{\sigma(t)}\otimes I_n) & -\alpha(L_{\sigma(t)}\otimes I_n)\end{bmatrix} \times \begin{bmatrix}x(t)\\ v(t)\end{bmatrix}, \quad t\in(t_{k-1}, t_k]. \tag{4}$$

At switching instance t_k, the time-delayed impulsive control signal as the input of an agent has the form:

$$\begin{bmatrix}\Delta x(t)\\ \Delta v(t)\end{bmatrix} = (I_{2N}\otimes B_k)\begin{bmatrix}x(t-\tau_k)\\ v(t-\tau_k)\end{bmatrix}, \quad t = t_k, \tag{5}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, B_k is an $n \times n$ control constant matrix, and τ_k is a delay constant satisfying: $\tau = \sup_k\{\tau_k\} \in [0, \infty)$, and there exists a positive integer i_0 such that $t_{k-i_0} < t_k - \tau \le t_{k-i_0+1}$ for any $k \ge i_0$. Let i_k, $k = 1, 2, \dots,$ be a set of nonnegative integers chosen in such a way that $t_{k-i_k} < t_k - \tau_k \le t_{k-i_k+1}$. Obviously, $1 \le i_0, i_k \le k$. Denote $\Delta_k = t_k - t_{k-1}$ and $\delta_{k-i_k+1} = t_{k-i_k+1} - t_k + \tau_k$, then we have $0 \le \delta_{k-i_k+1} < \Delta_{k-i_k+1}$. Besides, denote $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$, $v(t_k^+) = \lim_{h\to 0^+} v(t_k + h)$, and $v(t_k^-) = \lim_{h\to 0^+} v(t_k - h)$. Based on Eq. (4), it must be satisfied that $x(t_k^-) = x(t_k)$ and $v(t_k^-) = v(t_k)$. It should be noted that τ_k is called the computing delay, which reflects the controller's limit ability in processing information.

Therefore, with the time-delayed impulsive input signal described by Eq. (5), the closed-loop system of Eq. (4) has the form:

$$\begin{cases} \begin{bmatrix}\dot{x}(t)\\ \dot{v}(t)\end{bmatrix} = \begin{bmatrix}0\\ F(x,v)\end{bmatrix} + \begin{bmatrix}0 & I_N\otimes I_n\\ -\alpha(L_{\sigma(t)}\otimes I_n) & -\alpha(L_{\sigma(t)}\otimes I_n)\end{bmatrix} \\ \qquad \times \begin{bmatrix}x(t)\\ v(t)\end{bmatrix}, \quad t\in(t_{k-1}, t_k], \\ \begin{bmatrix}\Delta x(t)\\ \Delta v(t)\end{bmatrix} = \begin{bmatrix}I_N\otimes B_k & 0\\ 0 & I_N\otimes B_k\end{bmatrix}\begin{bmatrix}x(t-\tau_k)\\ v(t-\tau_k)\end{bmatrix}, \quad t = t_k, \\ \begin{bmatrix}x(t)\\ v(t)\end{bmatrix} = \begin{bmatrix}\varphi_1(t)\\ \varphi_2(t)\end{bmatrix}, \quad t\in[-\tau, 0], \end{cases} \tag{6}$$

where $\varphi_1(t), \varphi_2(t)$ are the arbitrary differentiable initial functions defined over $[-\tau, 0]$. Fortunately, the theory about existence and uniqueness of solutions for the system described by Eq. (6) has already been developed in [34].

4. Main results

In this section, the consensus problem of nonlinear multi-agent systems described by Eq. (6) with switching topologies is investigated. A new comparison principle method is successfully applied to the analysis of consensus, and some algebraic criteria are established. The following assumption is necessary to obtain the main results of the paper.

Assumption 1. For the nonlinear function $f(x, v)$, there exist two constants $l_1, l_2 > 0$, such that

$$\|f(x,v) - f(\bar{x},\bar{v})\| \le l_1\|x-\bar{x}\| + l_2\|v-\bar{v}\|. \tag{7}$$

Note that Assumption 1 is a Lipschitz condition, which is satisfied by, for example, all piecewise linear functions [3,30]. First of all, let $c = \max\{\sqrt{P}l_1, \sqrt{P}l_2 + 1\}$, $\beta = \max\{l_1, l_2\}$, $\eta = \max_{k\in\Im}\left\{\left\|\hat{L}_k\right\|\right\}$, $\gamma_k = e^{(\alpha\eta+c)\Delta_k}\|I_n + B_k\|$, $\mu_k^{(j)} = (\sqrt{P}\beta + \alpha\eta + 1)\Delta_j e^{(\alpha\eta+c)\Delta_j}\|B_k\|$ with $k - i_k + 2 \le j \le k$, $w_k^{(j)} = e^{(\alpha\eta+c)[\Delta_{j-i_j+1}-\delta_{j-i_j+1}]}\|B_k\|\left\|B_j\right\|$ with $k - i_k + 1 \le j \le k - 1$, and $\zeta_k = (\sqrt{P}\beta + \alpha\eta + 1)\delta_{k-i_k+1}e^{(\alpha\eta+c)\Delta_{k-i_k+1}}\|B_k\|$, then the main result of the paper is given by the following theorem:

Theorem 1. *Given a system described by Eq. (6), suppose that Assumption 1 is satisfied, let* $p_k = \min\{k - i_k, k - 1 - i_{k-1}, \dots, k - i_k + 1 - i_{k-i_k+1}\}$, *if there exist constants* $\lambda \in [0, 1)$ *and* Δ, *such that* $\rho(C_k) \le \lambda$, $\Delta_k \le \Delta$ *for any* $k \ge i_0$, *then the second-order consensus of the system can be achieved, i.e.* $\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = \lim_{t\to\infty}\left\|v_i(t) - v_j(t)\right\| = 0$. *Here* $C_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & 0 & \cdots & 1\\ \beta_k^{(1)} & \beta_k^{(2)} & \beta_k^{(3)} & \cdots & \beta_k^{(k-p_k)} \end{bmatrix}$, $\beta_k^{(j)} = \sum_{s\in S_{k,j}} w_k^{(s)} (1 \le j \le k - i_k - p_k)$, $\beta_k^{(j)} = \mu_k^{(j+p_k)} + \sum_{s\in S_{k,j}} w_k^{(s)} (k - i_k - p_k + 2 \le j \le k - p_k - 1)$, $\beta_k^{(k-i_k-p_k+1)} = \zeta_k + \sum_{s\in S_{k,k-i_k-p_k+1}} w_k^{(s)}$, *and* $\beta_k^{(k-p_k)} = \gamma_k + \mu_k^{(k)}$, *with the set* $S_{k,j}$ *depending on the indexes* k *and* j *directly related to the delay terms* τ_k.

Proof. Consider the following auxiliary function

$$V(t) = \|(M\otimes I_n)x(t)\| + \|(M\otimes I_n)v(t)\|, \tag{8}$$

where $M \in M_2^N$ is a $P \times N$ matrix. For some states $x(t)$ and $v(t)$, one may get $(M \otimes I_n)x(t) = 0$ or $(M \otimes I_n)v(t) = 0$, and in such situations, the function $V(t)$ is not differentiable. Hence, in order to calculate the increment of $V(t)$ with time, we adopt the Dini derivative of $V(t)$, and discuss the question in four cases.

Case 1: $(M \otimes I_n)x(t) \ne 0$ and $(M \otimes I_n)v(t) \ne 0$.

In this case, we have $D^+V(t) = \dot{V}(t)$. According to Eq. (6), it is calculated by

$$\begin{aligned} D^+V(t) &= \frac{x^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{x}(t)}{\|(M\otimes I_n)x(t)\|} + \frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{v}(t)}{\|(M\otimes I_n)v(t)\|} \\ &= \frac{x^T(t)(M\otimes I_n)^T(M\otimes I_n)v(t)}{\|(M\otimes I_n)x(t)\|} + \frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)F(x,v)}{\|(M\otimes I_n)v(t)\|} \end{aligned}$$

$$-\alpha\frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)(L_{\sigma_k}\otimes I_n)x(t)}{\|(M\otimes I_n)v(t)\|}$$
$$-\alpha\frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)(L_{\sigma_k}\otimes I_n)v(t)}{\|(M\otimes I_n)v(t)\|}. \tag{9}$$

By Lemma 1, there exists a $P\times P$ matrix $\hat{L}_{\sigma_k}$ such that $ML_{\sigma_k}=\hat{L}_{\sigma_k}M$. Hence, Eq. (9) is rewritten by

$$\begin{aligned}D^+V(t) &= \frac{x^T(t)(M\otimes I_n)^T(M\otimes I_n)v(t)}{\|(M\otimes I_n)x(t)\|}\\&\quad+\frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)F(x,v)}{\|(M\otimes I_n)v(t)\|}\\&\quad-\alpha\frac{v^T(t)(M\otimes I_n)^T(\hat{L}_{\sigma_k}\otimes I_n)(M\otimes I_n)x(t)}{\|(M\otimes I_n)v(t)\|}\\&\quad-\alpha\frac{v^T(t)(M\otimes I_n)^T(\hat{L}_{\sigma_k}\otimes I_n)(M\otimes I_n)v(t)}{\|(M\otimes I_n)v(t)\|},\end{aligned}\tag{10}$$

Therefore,

$$\begin{aligned}D^+V(t) &\le \frac{\left|x^T(t)(M\otimes I_n)^T(M\otimes I_n)v(t)\right|}{\|(M\otimes I_n)x(t)\|}\\&\quad+\frac{\left|v^T(t)(M\otimes I_n)^T(M\otimes I_n)F(x,v)\right|}{\|(M\otimes I_n)v(t)\|}\\&\quad+\alpha\frac{\left|v^T(t)(M\otimes I_n)^T(\hat{L}_{\sigma_k}\otimes I_n)(M\otimes I_n)x(t)\right|}{\|(M\otimes I_n)v(t)\|}\\&\quad+\alpha\frac{\left|v^T(t)(M\otimes I_n)^T(\hat{L}_{\sigma_k}\otimes I_n)(M\otimes I_n)v(t)\right|}{\|(M\otimes I_n)v(t)\|}\\&\le \|(M\otimes I_n)v(t)\|+\|(M\otimes I_n)F(x,v)\|\\&\quad+\alpha\eta\,\|(M\otimes I_n)x(t)\|+\alpha\eta\,\|(M\otimes I_n)v(t)\|\\&=\alpha\eta V(t)+\|(M\otimes I_n)v(t)\|+\|(M\otimes I_n)F(x,v)\|.\end{aligned}\tag{11}$$

Since $F(x,v)=\left[f^T(x_1,v_1),\dots,f^T(x_N,v_N)\right]^T$, we get

$$(M\otimes I_n)\cdot F(x,v)=\begin{bmatrix}f(x_{j_{11}},v_{j_{11}})-f(x_{j_{12}},v_{j_{12}})\\ \vdots\\ f(x_{j_{i1}},v_{j_{i1}})-f(x_{j_{i2}},v_{j_{i2}})\\ \vdots\\ f(x_{j_{P1}},v_{j_{P1}})-f(x_{j_{P2}},v_{j_{P2}})\end{bmatrix},\tag{12}$$

where j_{i1} and j_{i2} denote the column indexes of the first and second nonzero elements, respectively, in the ith row. Then

$$\begin{aligned}\|(M\otimes I_n)F(x,v)\| &= \sqrt{\sum_{i=1}^{P}\left\|f(x_{j_{i1}},v_{j_{i1}})-f(x_{j_{i2}},v_{j_{i2}})\right\|^2}\\&\le\sum_{i=1}^{P}\left\|f(x_{j_{i1}},v_{j_{i1}})-f(x_{j_{i2}},v_{j_{i2}})\right\|\\&\le l_1\left[\sum_{i=1}^{P}\left\|x_{j_{i1}}-x_{j_{i2}}\right\|\right]\\&\quad+l_2\left[\sum_{i=1}^{P}\left\|v_{j_{i1}}-v_{j_{i2}}\right\|\right]\\&\le l_1\sqrt{P\sum_{i=1}^{P}\left\|x_{j_{i1}}-x_{j_{i2}}\right\|^2}\\&\quad+l_2\sqrt{P\sum_{i=1}^{P}\left\|v_{j_{i1}}-v_{j_{i2}}\right\|^2}\\&=\sqrt{P}l_1\,\|(M\otimes I_n)x(t)\|\\&\quad+\sqrt{P}l_2\,\|(M\otimes I_n)v(t)\|.\end{aligned}\tag{13}$$

According to Eqs. (11) and (13), we obtain

$$\begin{aligned}D^+V(t) &\le \alpha\eta V(t)+\|(M\otimes I_n)v(t)\|+(\sqrt{P}l_1)\,\|(M\otimes I_n)x(t)\|\\&\quad+(\sqrt{P}l_2)\,\|(M\otimes I_n)v(t)\|\\&\le(\alpha\eta+c)V(t).\end{aligned}\tag{14}$$

Case 2: $(M\otimes I_n)x(t)=0$ and $(M\otimes I_n)v(t)\ne 0$.

In this case, $\|(M\otimes I_n)x(t)\|$ is not differentiable, denote $W_1(t)=\|(M\otimes I_n)x(t)\|^2$, then we have

$$\begin{aligned}W_1(t+h) &= W_1(t)+\dot{W}_1(t)h+\frac{1}{2}\ddot{W}_1(t)h^2+o(h^3)\\&=h^2\dot{x}^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{x}(t)+o(h^3).\end{aligned}\tag{15}$$

According to the definition of the Dini derivative, we get

$$\begin{aligned}D^+\left[\|(M\otimes I_n)x(t)\|\right] &= \lim_{h\to0^+}\sup\frac{\sqrt{W_1(t+h)}-\sqrt{W_1(t)}}{h}\\&=\lim_{h\to0^+}\sup\frac{\sqrt{W_1(t+h)}}{h}\\&=\lim_{h\to0^+}\sup\frac{\sqrt{h^2\dot{x}^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{x}(t)+o(h^3)}}{h}\\&=\sqrt{\dot{x}^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{x}(t)}\\&=\|(M\otimes I_n)v(t)\|.\end{aligned}\tag{16}$$

Hence,

$$\begin{aligned}D^+V(t) &= \lim_{h\to0^+}\sup\frac{V(t+h)-V(t)}{h}\\&\le D^+\left[\|(M\otimes I_n)x(t)\|\right]+\frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{v}(t)}{\|(M\otimes I_n)v(t)\|}\\&=\|(M\otimes I_n)v(t)\|+\frac{v^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{v}(t)}{\|(M\otimes I_n)v(t)\|}.\end{aligned}\tag{17}$$

Based on Eqs. (11) and (17), it implies

$$D^+V(t)\le(\alpha\eta+c)V(t).\tag{18}$$

Case 3: $(M\otimes I_n)x(t)\ne 0$ and $(M\otimes I_n)v(t)=0$.

In this case, $\|(M\otimes I_n)v(t)\|$ is also not differentiable, denote $W_2(t)=\|(M\otimes I_n)v(t)\|^2$, then we have

$$\begin{aligned}W_2(t+h) &= W_2(t)+\dot{W}_2(t)h+\frac{1}{2}\ddot{W}_2(t)h^2+o(h^3)\\&=h^2\dot{v}^T(t)(M\otimes I_n)^T(M\otimes I_n)\dot{v}(t)+o(h^3).\end{aligned}\tag{19}$$

Similar to Eq. (16), we obtain

$$D^+\left[\|(M\otimes I_n)v(t)\|\right]=\|(M\otimes I_n)\dot{v}(t)\|.\tag{20}$$

Therefore,

$$\begin{aligned}D^+V(t) &= \lim_{h\to0^+}\sup\frac{V(t+h)-V(t)}{h}\\&\le\frac{x^T(t)(M\otimes I_n)^T(M\otimes I_n)v(t)}{\|(M\otimes I_n)v(t)\|}+D^+\left[\|(M\otimes I_n)v(t)\|\right]\\&=\frac{x^T(t)(M\otimes I_n)^T(M\otimes I_n)v(t)}{\|(M\otimes I_n)v(t)\|}+\|(M\otimes I_n)\dot{v}(t)\|.\end{aligned}\tag{21}$$

According to Eqs. (13) and (21), we obtain

$$D^+V(t) \le (\alpha\eta + c)V(t). \tag{22}$$

Case 4: $(M \otimes I_n)x(t) = 0$ and $(M \otimes I_n)v(t) = 0$.

In this case, we know that $x_i(t) = x_j(t)$, $v_i(t) = v_j(t)$ for each $i \in I$. According to Eq. (6), we get $x_i(t') = x_j(t')$, $v_i(t') = v_j(t')$ for $t' \in [t, \infty)$, and thus, $V(t') = V(t) = 0$, for $t' \in [t, \infty)$, and $D^+V(t) = 0$.

Then for each $t \in (t_{k-1}, t_k)$, we get

$$D^+V(t) \le (\alpha\eta + c)V(t). \tag{23}$$

By the comparison lemma [32], the following inequality holds:

$$V(t) \le e^{(\alpha\eta+c)(t-t_{k-1})}V(t_{k-1}^+) \le e^{(\alpha\eta+c)\Delta}V(t_{k-1}^+). \tag{24}$$

Next, consider the term $V(t_k^+)$, by Eq. (6), we have

$$\begin{aligned} x(t_k^+) &= (I_N \otimes B_k)\left[x(t_k - \tau_k) - x(t_k)\right] \\ &\quad + \left[I_N \otimes (I_n + B_k)\right]x(t_k), \end{aligned} \tag{25}$$

$$\begin{aligned} v(t_k^+) &= (I_N \otimes B_k)\left[v(t_k - \tau_k) - v(t_k)\right] \\ &\quad + \left[I_N \otimes (I_n + B_k)\right]v(t_k). \end{aligned} \tag{26}$$

By applying the Mean Value Theorem in $[t_k - \tau_k, t_{k-i_k+1}]$, we obtain

$$x(t_k - \tau_k) - x(t_{k-i_k+1}) = -\delta_{k-i_k+1} \cdot v(t), \tag{27}$$

where $t \in [t_k - \tau_k, t_{k-i_k+1}]$. Thus, from Eq. (25), we have

$$\begin{aligned} (M \otimes I_n)x(t_k^+) &= (M \otimes I_n)(I_N \otimes B_k)\left[x(t_{k-i_k+1}) - x(t_k)\right] \\ &\quad - \delta_{k-i_k+1}(M \otimes I_n)(I_N \otimes B_k)v(t) \\ &\quad + (M \otimes I_n)\left[I_N \otimes (I_n + B_k)\right]x(t_k) \\ &= -(I_P \otimes B_k)(M \otimes I_n)\left[x(t_k) - x(t_{k-i_k+1})\right] \\ &\quad - \delta_{k-i_k+1}(I_P \otimes B_k)(M \otimes I_n)v(t) \\ &\quad + \left[I_P \otimes (I_n + B_k)\right](M \otimes I_n)x(t_k). \end{aligned} \tag{28}$$

Consider the term $(M \otimes I_n)\left[x(t_k) - x(t_{k-i_k+1})\right]$,

$$\begin{aligned} &(M \otimes I_n)\left[x(t_k) - x(t_{k-i_k+1})\right] \\ &= (M \otimes I_n)\sum_{j=k-i_k+2}^{k}\left[x(t_j) - x(t_{j-1})\right] \\ &= (M \otimes I_n)\sum_{j=k-i_k+2}^{k}\left[x(t_j) - x(t_{j-1}^+) + x(t_{j-1}^+) - x(t_{j-1})\right] \\ &= (M \otimes I_n)\sum_{j=k-i_k+2}^{k}\left[x(t_j) - x(t_{j-1}^+)\right] \\ &\quad + (M \otimes I_n)\sum_{j=k-i_k+1}^{k-1}\left[x(t_j^+) - x(t_j)\right] \\ &= (M \otimes I_n)\sum_{j=k-i_k+2}^{k}\left[x(t_j) - x(t_{j-1}^+)\right] \\ &\quad + (M \otimes I_n)\sum_{j=k-i_k+1}^{k-1}(I_N \otimes B_j)x(t_j - \tau_j). \end{aligned} \tag{29}$$

Thus, we can rewrite Eq. (28) by

$$\begin{aligned} (M \otimes I_n)x(t_k^+) &= -(I_P \otimes B_k)(M \otimes I_n)\sum_{j=k-i_k+2}^{k}\left[x(t_j) - x(t_{j-1}^+)\right] \\ &\quad + \left[I_P \otimes (I_n + B_k)\right](M \otimes I_n)x(t_k) \\ &\quad - (I_P \otimes B_k)(M \otimes I_n)\sum_{j=k-i_k+1}^{k-1}(I_N \otimes B_j)x(t_j - \tau_j) \\ &\quad - \delta_{k-i_k+1}(I_P \otimes B_k)(M \otimes I_n)v(t). \end{aligned} \tag{30}$$

Moreover,

$$\begin{aligned} &\left\|(M \otimes I_n)x(t_k^+)\right\| \\ &\le \|I_P \otimes B_k\|\sum_{j=k-i_k+2}^{k}\left\|(M \otimes I_n)\left[x(t_j) - x(t_{j-1}^+)\right]\right\| \\ &\quad + \delta_{k-i_k+1}\|I_P \otimes B_k\| \cdot \|(M \otimes I_n)v(t)\| \\ &\quad + \|I_P \otimes B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|(I_P \otimes B_j)\right\| \cdot \left\|(M \otimes I_n)x(t_j - \tau_j)\right\| \\ &\quad + \|I_P \otimes (I_n + B_k)\| \cdot \|(M \otimes I_n)x(t_k)\|. \end{aligned} \tag{31}$$

Considering the term $\left\|(M \otimes I_n)\left[x(t_j) - x(t_{j-1}^+)\right]\right\|$, we have

$$\begin{aligned} \left\|(M \otimes I_n)\left[x(t_j) - x(t_{j-1}^+)\right]\right\| &= \left\|(M \otimes I_n)\int_{t_{j-1}^+}^{t_j} v(s)ds\right\| \\ &\le \int_{t_{j-1}^+}^{t_j}\|(M \otimes I_n)v(s)\|\,ds \\ &\le \int_{t_{j-1}^+}^{t_j} V(s)ds \\ &\le \int_{t_{j-1}^+}^{t_j} e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)ds \\ &= \Delta_j e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+). \end{aligned} \tag{32}$$

Thus,

$$\begin{aligned} \left\|(M \otimes I_n)x(t_k^+)\right\| &\le \|I_P \otimes B_k\|\sum_{j=k-i_k+2}^{k}\Delta_j e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+) \\ &\quad + \delta_{k-i_k+1}\|I_P \otimes B_k\| \cdot \|(M \otimes I_n)v(t)\| \\ &\quad + \|I_P \otimes B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|(I_P \otimes B_j)\right\| \\ &\quad \cdot \left\|(M \otimes I_n)x(t_j - \tau_j)\right\| \\ &\quad + \|I_P \otimes (I_n + B_k)\| \cdot \|(M \otimes I_n)x(t_k)\| \\ &= \|B_k\|\sum_{j=k-i_k+2}^{k}\Delta_j e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+) \\ &\quad + \delta_{k-i_k+1}\|B_k\| \cdot \|(M \otimes I_n)v(t)\| \\ &\quad + \|B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|B_j\right\| \cdot \left\|(M \otimes I_n)x(t_j - \tau_j)\right\| \\ &\quad + \|I_n + B_k\| \cdot \|(M \otimes I_n)x(t_k)\|. \end{aligned} \tag{33}$$

According to the Mean Value Theorem, we get

$$\begin{aligned} v(t_k - \tau_k) - v(t_{k-i_k+1}) &= -\delta_{k-i_k+1} \cdot \dot{v}(t') \\ &= -\delta_{k-i_k+1} \cdot F(x(t'), v(t')) \\ &\quad + \alpha\delta_{k-i_k+1}(L_{\sigma_{k-i_k+1}} \otimes I_n)x(t') \\ &\quad + \alpha\delta_{k-i_k+1}(L_{\sigma_{k-i_k+1}} \otimes I_n)v(t') \end{aligned} \tag{34}$$

where $t' \in [t_k - \tau_k, t_{k-i_k+1}]$. Thus, from Eqs. (26) and (34), we have

$$\begin{aligned} (M \otimes I_n)v(t_k^+) &= (M \otimes I_n)(I_N \otimes B_k)\left[v(t_{k-i_k+1}) - v(t_k)\right] \\ &\quad - \delta_{k-i_k+1}(M \otimes I_n)(I_N \otimes B_k)\dot{v}(t') \end{aligned}$$

$$
\begin{aligned}
&+(M\otimes I_n)\left[I_N\otimes(I_n+B_k)\right]v(t_k)\\
=\;&(M\otimes I_n)(I_N\otimes B_k)\left[v(t_{k-i_k+1})-v(t_k)\right]\\
&-\delta_{k-i_k+1}(M\otimes I_n)(I_N\otimes B_k)F(x(t'),v(t'))\\
&+\alpha\delta_{k-i_k+1}(M\otimes I_n)(I_N\otimes B_k)(L_{\sigma_{k-i_k+1}}\otimes I_n)x(t')\\
&+(M\otimes I_n)\left[I_N\otimes(I_n+B_k)\right]v(t_k)\\
&+\alpha\delta_{k-i_k+1}(M\otimes I_n)(I_N\otimes B_k)(L_{\sigma_{k-i_k+1}}\otimes I_n)v(t')\\
=\;&-(I_P\otimes B_k)(M\otimes I_n)\left[v(t_k)-v(t_{k-i_k+1})\right]\\
&-\delta_{k-i_k+1}(I_P\otimes B_k)(M\otimes I_n)F(x(t'),v(t'))\\
&+\alpha\delta_{k-i_k+1}(I_P\otimes B_k)(\hat{L}_{\sigma_{k-i_k+1}}\otimes I_n)(M\otimes I_n)x(t')\\
&+\left[I_P\otimes(I_n+B_k)\right](M\otimes I_n)v(t_k)+\alpha\delta_{k-i_k+1}\\
&\times(I_P\otimes B_k)(\hat{L}_{\sigma_{k-i_k+1}}\otimes I_n)(M\otimes I_n)v(t'). \qquad (35)
\end{aligned}
$$

Similar to Eq. (29), we obtain

$$
\begin{aligned}
&(M\otimes I_n)v(t_k^+)\\
&=-(I_P\otimes B_k)(M\otimes I_n)\left\{\sum_{j=k-i_k+2}^{k}\left[v(t_j)-v(t_{j-1}^+)\right]\right.\\
&\quad\left.+\sum_{j=k-i_k+1}^{k-1}(I_N\otimes B_j)v(t_j-\tau_j)\right\}\\
&\quad-\delta_{k-i_k+1}(I_P\otimes B_k)(M\otimes I_n)F(x(t'),v(t'))\\
&\quad+\alpha\delta_{k-i_k+1}(I_P\otimes B_k)(\hat{L}_{\sigma_{k-i_k+1}}\otimes I_n)(M\otimes I_n)x(t')\\
&\quad+\alpha\delta_{k-i_k+1}(I_P\otimes B_k)(\hat{L}_{\sigma_{k-i_k+1}}\otimes I_n)(M\otimes I_n)v(t')\\
&\quad+\left[I_P\otimes(I_n+B_k)\right](M\otimes I_n)v(t_k). \qquad (36)
\end{aligned}
$$

Consider the term $\left\|(M\otimes I_n)\left[x(t_j)-x(t_{j-1}^+)\right]\right\|$, from Eq. (6), we have

$$
\begin{aligned}
&\left\|(M\otimes I_n)\left[v(t_j)-v(t_{j-1}^+)\right]\right\|\\
&\le\int_{t_{j-1}^+}^{t_j}\|(M\otimes I_n)F(x(s),v(s))\|\,ds\\
&\quad+\alpha\eta\int_{t_{j-1}^+}^{t_j}\|(M\otimes I_n)\left[x(s)+v(s)\right]\|\,ds\\
&\le\int_{t_{j-1}^+}^{t_j}\sqrt{P}\beta V(s)ds+\alpha\eta\int_{t_{j-1}^+}^{t_j}V(s)ds\\
&\le(\sqrt{P}\beta+\alpha\eta)\int_{t_{j-1}^+}^{t_j}e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)ds\\
&=\Delta_j(\sqrt{P}\beta+\alpha\eta)e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+). \qquad (37)
\end{aligned}
$$

Hence, we can rewrite Eq. (36) by

$$
\begin{aligned}
&\left\|(M\otimes I_n)v(t_k^+)\right\|\\
&\le\|(I_P\otimes B_k)\|\sum_{j=k-i_k+2}^{k}\Delta_j(\sqrt{P}\beta+\alpha\eta)e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)\\
&\quad+\|[I_P\otimes(I_n+B_k)]\|\cdot\|(M\otimes I_n)v(t_k)\|\\
&\quad+\alpha\eta\delta_{k-i_k+1}\|(I_P\otimes B_k)\|\left[\left\|(M\otimes I_n)x(t')\right\|\right.\\
&\quad\left.+\left\|(M\otimes I_n)v(t')\right\|\right]+\delta_{k-i_k+1}\|(I_P\otimes B_k)\|\\
&\quad\times\left\|(M\otimes I_n)F(x(t'),v(t'))\right\|+\|(I_P\otimes B_k)\|\\
&\quad\times\sum_{j=k-i_k+1}^{k-1}\left\|I_N\otimes B_j\right\|\left\|(M\otimes I_n)v(t_j-\tau_j)\right\|\\
&\le\|B_k\|\sum_{j=k-i_k+2}^{k}\Delta_j(\sqrt{P}\beta+\alpha\eta)e^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)\\
&\quad+\|I_n+B_k\|\cdot\|(M\otimes I_n)v(t_k)\|\\
&\quad+\alpha\eta\delta_{k-i_k+1}\|B_k\|\left[\left\|(M\otimes I_n)x(t')\right\|+\left\|(M\otimes I_n)v(t')\right\|\right]\\
&\quad+\|B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|B_j\right\|\left\|(M\otimes I_n)v(t_j-\tau_j)\right\|+\delta_{k-i_k+1}\sqrt{P}\beta\\
&\quad\times\|B_k\|\left[\left\|(M\otimes I_n)x(t')\right\|+\left\|(M\otimes I_n)v(t')\right\|\right]. \qquad (38)
\end{aligned}
$$

From Eqs. (33) and (38), we obtain

$$
\begin{aligned}
V(t_k^+)\le\;&(\sqrt{P}\beta+\alpha\eta+1)\|B_k\|\sum_{j=k-i_k+2}^{k}\Delta_je^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)\\
&+\|B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|B_j\right\|\cdot V(t_j-\tau_j)\\
&+\delta_{k-i_k+1}\|B_k\|\cdot\|(M\otimes I_n)v(t)\|\\
&+(\sqrt{P}\beta+\alpha\eta)\delta_{k-i_k+1}\|B_k\|\cdot V(t')+\|I_n+B_k\|\cdot V(t_k)\\
\le\;&(\sqrt{P}\beta+\alpha\eta+1)\|B_k\|\sum_{j=k-i_k+2}^{k}\Delta_je^{(\alpha\eta+c)\Delta_j}V(t_{j-1}^+)\\
&+\|I_n+B_k\|\,e^{(\alpha\eta+c)\Delta_k}V(t_{k-1}^+)\\
&+\delta_{k-i_k+1}\|B_k\|\,e^{(\alpha\eta+c)\Delta_{k-i_k+1}}V(t_{k-i_k}^+)\\
&+\|B_k\|\sum_{j=k-i_k+1}^{k-1}\left\|B_j\right\|\cdot e^{(\alpha\eta+c)\left[\Delta_{j-i_j+1}-\delta_{j-i_j+1}\right]}V(t_{j-i_j}^+)\\
&+(\sqrt{P}\beta+\alpha\eta)\delta_{k-i_k+1}\|B_k\|\,e^{(\alpha\eta+c)\Delta_{k-i_k+1}}V(t_{k-i_k}^+). \qquad (39)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
V(t_k^+)\le\;&\gamma_kV(t_{k-1}^+)+\sum_{j=k-i_k+2}^{k}\mu_k^{(j)}V(t_{j-1}^+)\\
&+\sum_{j=k-i_k+1}^{k-1}w_k^{(j)}V(t_{j-i_j}^+)+\zeta_kV(t_{k-i_k}^+). \qquad (40)
\end{aligned}
$$

Denote $W(k)=\left[V(t_k^+),V(t_{k+1}^+),\ldots,V(t_{k+i_0-1}^+)\right]^T\in R^{i_0}$, then, by Eq. (40), it follows

$$
\begin{aligned}
W(k-i_0+1)&\le\begin{bmatrix}A_k & B_k\\ 0 & C_k\end{bmatrix}\begin{bmatrix}V(t_{k-i_0}^+)\\ \vdots\\ V(t_{p_k-1}^+)\\ V(t_{p_k}^+)\\ \vdots\\ V(t_{k-1}^+)\end{bmatrix}\\
&=\begin{bmatrix}A_k & B_k\\ 0 & C_k\end{bmatrix}W(k-i_0), \qquad (41)
\end{aligned}
$$

where $k\ge i_0$, $A_k=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \ldots&\ldots&\ldots&\ddots&0\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix}\in R^{(i_0-k+p_k)\times(i_0-k+p_k)}$,

$B_k=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\ \ldots&\ldots&\ldots&\ldots\\1&0&\cdots&0\end{bmatrix}$, and the inequality (41) holds componentwise. Denote the comparison system by

$$\begin{cases} Z(k+1) = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix} Z(k), \quad k \geq i_0 \\ Z(i_0) = W(0). \end{cases} \tag{42}$$

Then, by the comparison principle [35], we get

$$W(k-i_0) \leq Z(k), \quad \forall k \geq i_0. \tag{43}$$

According to the definitions of A_k and B_k, we have

$$\rho \left\{ \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix} \right\} = \rho(C_k) \leq \lambda, \quad \forall k \geq i_0. \tag{44}$$

Hence,

$$\begin{aligned} \lim_{k\to\infty} V(t_k^+) &\leq \lim_{k\to\infty} \|W(k-i_0)\| \\ &\leq \lim_{k\to\infty} \|Z(k)\| = 0. \end{aligned} \tag{45}$$

By Eqs. (24) and (45), we have

$$\lim_{t\to\infty} V(t) \leq \lim_{k\to\infty} e^{(\alpha\eta+c)\Delta} V(t_k^+) = 0. \tag{46}$$

Thus,

$$\lim_{t\to\infty} \|(M \otimes I_n)x(t)\| = \lim_{t\to\infty} \|(M \otimes I_n)v(t)\| = 0. \tag{47}$$

From the definition of M_2^N, we can conclude that

$$\begin{aligned} &\lim_{t\to\infty} \left\|x_i(t) - x_j(t)\right\| = \lim_{t\to\infty} \left\|v_i(t) - v_j(t)\right\| = 0, \\ &\forall i, j \in I. \end{aligned} \tag{48}$$

Thus the second-order consensus of the system described by Eq. (6) is achieved.

Remark 1. It should be noted that the system described by Eq. (6) has a more general network topology than those considered in the literature [5,6,12–14,22,26]. Actually, in Theorem 1, we do not assume that the network topology is strongly connected or contains a directed spanning tree. Meanwhile, the weight matrix of the coupling graph is also not assumed to be nonnegative, which implies that the agents in the system can both cooperate and compete with each other. Generally, it is necessary to have a directed spanning tree only when considering the consensus problem on cooperative networks without any extra controller. In such a situation, the network does not have a directed spanning tree means that the network must have at least two strongly connected components between which there is no information exchange, and thus the agents in the network cannot achieve consensus spontaneously. In this paper, we consider the extra competition relationships between agents and the impulsive input, as a result, a directed spanning tree established by cooperative links is not a necessary condition of consensus any longer. First, the competition relationships may increase the chance of information exchange between the agents from different strongly connected components (established by cooperative links). Second, the impulsive control scheme guarantees the errors between the states of agents from different strongly connected components converge to zero asymptotically, so that all the agents in the system can achieve consensus even when there lacks a directed spanning tree in the network.

Remark 2. In Theorem 1, by the comparison principle, we solve the second-order consensus problem asymptotically. However, there is another important problem: how quickly the agents can reach consensus? In order to solve such consensus rate problem, we will adopt the definition of exponentially consensus introduced by Yu et al. [15]. The multi-agent system is said to be exponentially consensus if there are parameters $\xi > 0, b > 0, T > 0$ and $0 < q < 1$ such that $\left\|x_i(t) - x_j(t)\right\| \leq bq^{\xi t}$, $\left\|v_i(t) - v_j(t)\right\| \leq bq^{\xi t}$ for all $t > T, i, j \in I$. It can be proved that the system described by Eq. (6) is exponentially consensus. In fact, based on Eqs. (42) and (44), we can conclude that there exists $0 < q < 1$ such that $\|Z(k)\| \leq q^{k-i_0} \|Z(i_0)\|$ for $k \geq i_0$. Hence, $\|W(k-i_0)\| \leq q^{k-i_0} \|W(0)\|$. According to the definition of $W(k)$, we have $V(t_k^+) \leq q^{k-i_0} \|W(0)\|$. For any t, there exists k such that $t \in (t_k, t_{k+1}]$, and by (24), we get $V(t) \leq e^{(\alpha\eta+c)\Delta} V(t_k^+) \leq e^{(\alpha\eta+c)\Delta} \|W(0)\| q^{k-i_0}$. Since $\Delta_k \leq \Delta$, we have $k+1 \geq \frac{t}{\Delta}$, which implies $V(t) \leq e^{(\alpha\eta+c)\Delta} \|W(0)\| q^{\frac{t}{\Delta}-1-i_0}$. Denote $\xi = \frac{1}{\Delta}$, $b = e^{(\alpha\eta+c)\Delta} q^{-(1+i_0)} \|W(0)\|$, $T = t_{i_0}$, then the proof is finished.

Remark 3. In Theorem 1, it is assumed that there is a constant Δ, such that $t_{k+1} - t_k \leq \Delta$, for any k, which means that the switching topologies between agents must be happened frequently. In fact, this is a necessary condition to achieve consensus: the impulsive control signal is added to an agent only at switching time t_k, as a result, if $\lim_{k\to\infty}(t_{k+1} - t_k) = \infty$, the impulsive control signal will be too infrequent to make the system achieve consensus.

Note that Theorem 1 gives the consensus criteria for the general case, and we need to check the eigenvalues of an infinite but countable number of matrices C_k. In the practical situation, the time-delayed impulsive control signal can be designed by the constant matrix B_k and the delay term τ_k such that the agents can achieve consensus in the system described by Eq. (6). Hence, variation in the delay term τ_k appearing in the impulses is not significant. With the reason that multi-agent systems are always distributed computer control systems in practice, i.e., Δ_k is kept unchanged, we consider the system with equidistant switching and a fixed delay term in the following corollary.

Corollary 1. *Given the system described by Eq. (6), suppose that Assumption 1 holds, and $\Delta_k = \Delta$, $\tau_k = \tau$ for any k, if there exists a constant $\lambda \in [0, 1)$, such that $\rho(C_k) \leq \lambda$, for any $k \geq i_0$, then the second-order consensus of the system is achieved, i.e. $\lim_{t\to\infty} \left\|x_i(t) - x_j(t)\right\| = \lim_{t\to\infty} \left\|v_i(t) - v_j(t)\right\| = 0$, where* $C_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta_k^{(1)} & \beta_k^{(2)} & \beta_k^{(3)} & \cdots & \beta_k^{(2i_0-1)} \end{bmatrix} \in R^{(2i_0-1)\times(2i_0-1)}$, $\beta_k^{(j)} = w_k^{(k+j-i_0)} (1 \leq j \leq i_0 - 1)$, $\beta_k^{(i_0)} = \zeta_k$, $\beta_k^{(j)} = \mu_k^{(j+k-2i_0+1)} (i_0 + 1 \leq j \leq 2i_0 - 2)$, $\beta_k^{(2i_0-1)} = \gamma_k + \mu_k^{(k)}$, $\gamma_k = e^{(\alpha\eta+c)\Delta} \|I_n + B_k\|$, $\mu_k^{(j)} = (\sqrt{P}\beta + \alpha\eta + 1)\Delta e^{(\alpha\eta+c)\Delta} \|B_k\|$ *with* $k - i_0 + 2 \leq j \leq k$, $w_k^{(j)} = e^{(\alpha\eta+c)[\Delta-\delta]} \|B_k\| \left\|B_j\right\|$ *with* $k - i_0 + 1 \leq j \leq k-1$, *and* $\zeta_k = (\sqrt{P}\beta + \alpha\eta + 1)\delta e^{(\alpha\eta+c)\Delta} \|B_k\|$.

Theorem 1 and Corollary 1 imply that, as τ_k increases, the magnitude of the impulses $\|B_k\|$ must be small enough in order to achieve the consensus. So there is a tradeoff between τ_k and $\|B_k\|$. Furthermore, for some large τ_k, one may not be able to find a suitable matrix B_k that can maintain the consensus of the system. Therefore, in the following corollary, we consider the system with short time-delay, i.e. $t_{k-1} < t_k - \tau_k \leq t_k$ for any k.

Corollary 2. *Given the system described by Eq. (6), suppose that Assumption 1 holds, and $t_{k-1} < t_k - \tau_k \leq t_k$ for any k, if there exists a constant $\lambda \in [0, 1)$, such that $C_k \leq \lambda$, for any $k \geq 1$, then the second-order consensus of the system is achieved, i.e. $\lim_{t\to\infty} \left\|x_i(t) - x_j(t)\right\| = \lim_{t\to\infty} \left\|v_i(t) - v_j(t)\right\| = 0$, where* $C_k = \left[\left(\sqrt{P}\beta + \alpha\eta + 1\right) \tau_k \|B_k\| + \|I_n + B_k\|\right] e^{(\alpha\eta+c)\Delta_k}$.

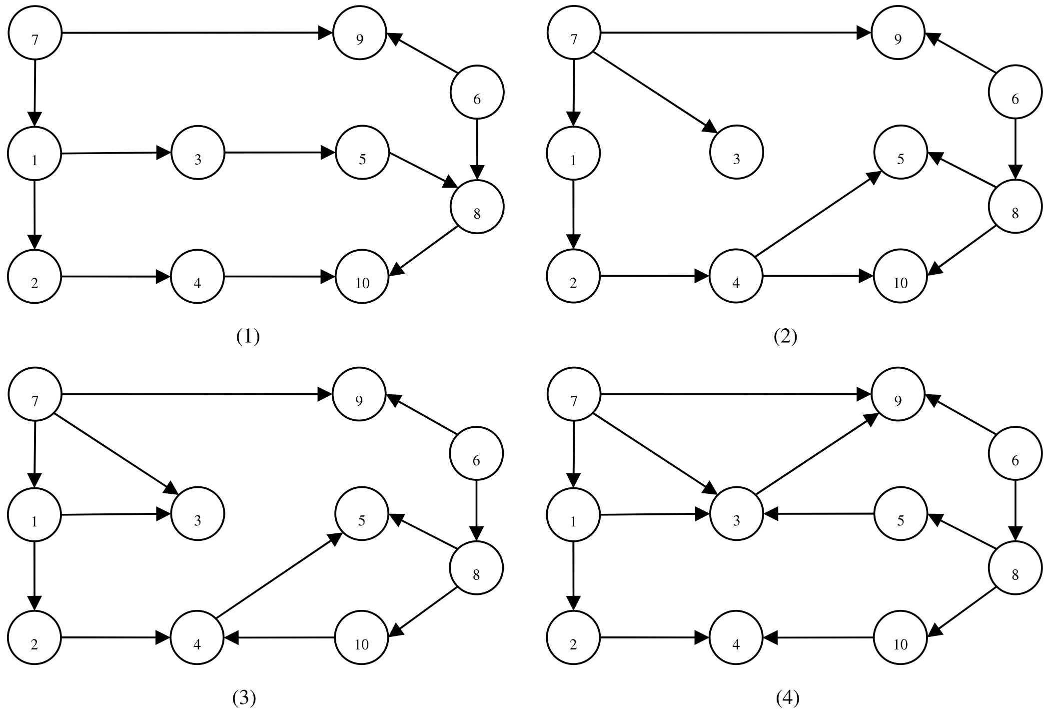

Fig. 1. Four different interaction topologies of ten agents.

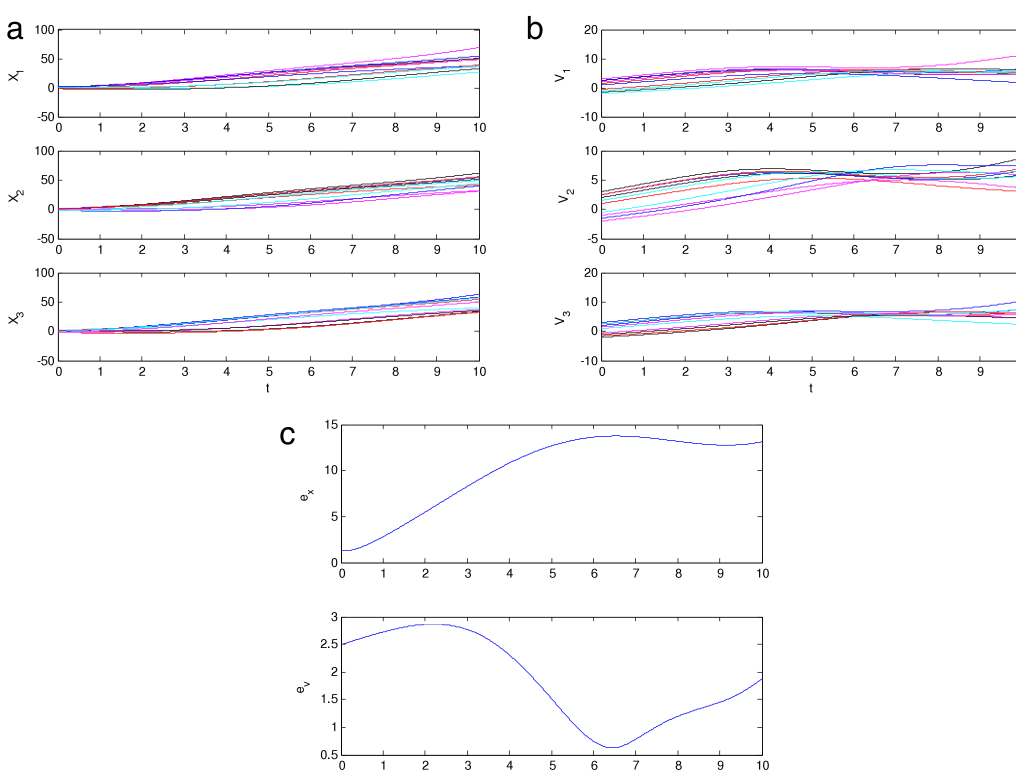

Fig. 2. (a) The evolution of $x(t)$ with switching topologies. (b) The evolution of $v(t)$ with switching topologies. (c) The evolution of the total error with switching topologies.

5. Numerical simulation

In this section, two numerical examples are provided in order to validate the proposed theoretical results.

Example 1. Consider 10 agents with switching topologies represented by Fig. 1, and the agents are governed by the following nonlinear intrinsic dynamics

$$\dot{v}_i(t) = \cos(0.1x_i) + \tanh(0.1v_i) + \alpha \sum_{j\in N_i(t)} a_{ij}[(x_j - x_i) + (v_j - v_i)], \quad \forall i \in I, \tag{49}$$

where $x_i(t), v_i(t) \in R^3$, $\tanh(\cdot)$ is the hyperbolic tangent function, and defined component-wise, and the coupling strength is chosen as $\alpha = 0.01$. Here, the nonlinear function f in Eq. (49) satisfies Assumption 1, and $l_1 = l_2 = 0.1$. Initially, the agents are randomly distributed in a three-dimensional space cube $[-2, 2] \times [-2, 2] \times [-2, 2]$ with their velocities limited in $[-3, 3] \times [-3, 3] \times [-3, 3]$. The total error of the multi-agent system with respect to agent 1 is defined by

$$
\begin{aligned}
e_x &\triangleq \frac{1}{10}\sum_{i=1}^{3}\sqrt{\sum_{j=1}^{10}\left[x_{1i}(t) - x_{ji}(t)\right]^2},\\
e_v &\triangleq \frac{1}{10}\sum_{i=1}^{3}\sqrt{\sum_{j=1}^{10}\left[v_{1i}(t) - v_{ji}(t)\right]^2}.
\end{aligned}
\tag{50}
$$

Here, the interaction topology of ten agents is presented in Fig. 1.

Moreover, the weight matrices are set to be

$$
\begin{aligned}
A_1 &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix},\\
A_3 &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix},\\
A_2 &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix},\\
A_4 &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\end{aligned}
\tag{51}
$$

As we can see, Eq. (51) implies that the multi-agent system considered here also include both cooperation and competition relationships. Suppose that the values of the switching signal $\sigma(t)$ are chosen randomly from $\{1, 2, 3, 4\}$. Without the time-delayed impulsive control, the evolutions of $x(t)$, $v(t)$, and $e(t)$ of the agents are shown in Fig. 2, and the multi-agent system also cannot achieve consensus.

Next, we will provide a time-delayed impulsive control scheme to solve the above consensus problem. First of all, the matrices M and G_M are set to be

$$
\begin{aligned}
M &= \begin{bmatrix}
1 & -1 & & & \\
 & 1 & -1 & & \\
 & & \ddots & & \\
 & & & 1 & -1
\end{bmatrix} \in R^{9\times 10},\\
G_M &= \begin{bmatrix}
1 & 1 & 1 & \cdots & 1\\
0 & 1 & 1 & \cdots & 1\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 1\\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} \in R^{10\times 9}.
\end{aligned}
\tag{52}
$$

Next, we consider the time-delayed impulsive control with the parameters set the same: $\Delta_k = 0.5$ for any $k \geq 2$, $t_1 = 5$, and the random delay term $\tau_k \in [0, 0.5]$. Similarly, we have $t_{k-1} < t_k - \tau_k \leq t_k$ for any k. From Corollary 2, we get the feasible solution of B_{A_k} as follows:

$$
\begin{aligned}
B_{A_1} &= \begin{bmatrix}
-1.4561 & 0.012 & -0.0617\\
0.012 & -1.4732 & 0.0038\\
-.0.0617 & 0.0038 & -1.3897
\end{bmatrix},\\
B_{A_2} &= \begin{bmatrix}
-1.3832 & 0.015 & -0.097\\
0.015 & -1.4156 & 0.047\\
-0.097 & 0.047 & -1.4325
\end{bmatrix},\\
B_{A_3} &= \begin{bmatrix}
-1.4639 & 0.02 & -0.013\\
0.02 & -1.3657 & 0.005\\
-0.013 & 0.005 & -1.3946
\end{bmatrix},\\
B_{A_4} &= \begin{bmatrix}
-1.4594 & 0.021 & -0.018\\
0.021 & -1.4156 & 0.006\\
-0.018 & 0.006 & -1.4325
\end{bmatrix}
\end{aligned}
\tag{53}
$$

and $C_{A_1} = 0.9768$, $C_{A_2} = 0.9971$, $C_{A_3} = 0.9223$, $C_{A_4} = 0.9345$. Consequently, the agents in the system described by Eq. (49) can achieve consensus. Correspondingly, the evolutions of $x(t)$, $v(t)$ and $e(t)$ of the agents with the impulsive control matrix B_{A_k} are shown in Fig. 3.

Example 2. Consider the same multi-agent system in Example 1 with the same initial conditions and other parameters. If we set $\Delta_k = 0.1$, $\tau_k = 0.15$ for any $k \geq 2$, then $t_{k-2} < t_k - \tau_k \leq t_{k-1}$, i.e., $i_0 = 2$, and $C_{A_k} \in R^{3\times 3}$. By Corollary 1, we design the time-delayed impulsive controllers to solve the consensus problem of system (49). Furthermore, the feasible solution of B_{A_k} can be obtained as follows:

$$
\begin{aligned}
B_{A_1} &= \begin{bmatrix}
-0.5036 & 0.013 & -0.064\\
0.013 & -0.4998 & -0.01\\
-.0.064 & -0.01 & -.0.5007
\end{bmatrix},\\
B_{A_2} &= \begin{bmatrix}
-0.6203 & 0.025 & -0.007\\
0.025 & -0.5673 & -0.019\\
-0.007 & -0.019 & -0.5547
\end{bmatrix},\\
B_{A_3} &= \begin{bmatrix}
-0.4963 & -0.022 & 0.017\\
-0.022 & -0.5857 & 0.02\\
0.017 & 0.02 & -0.4945
\end{bmatrix},\\
B_{A_4} &= \begin{bmatrix}
-0.4893 & -0.011 & 0.027\\
-0.011 & -0.5198 & 0.034\\
0.027 & 0.0034 & -0.6023
\end{bmatrix}
\end{aligned}
\tag{54}
$$

and $\rho(C_{A_1}) = 0.9566$, $\rho(C_{A_2}) = 0.9697$, $\rho(C_{A_3}) = 0.9624$, $\rho(C_{A_4}) = 0.9676$. Thus, according to Corollary 1, the agents can achieve consensus by using the impulsive control matrix B_{A_k}. The

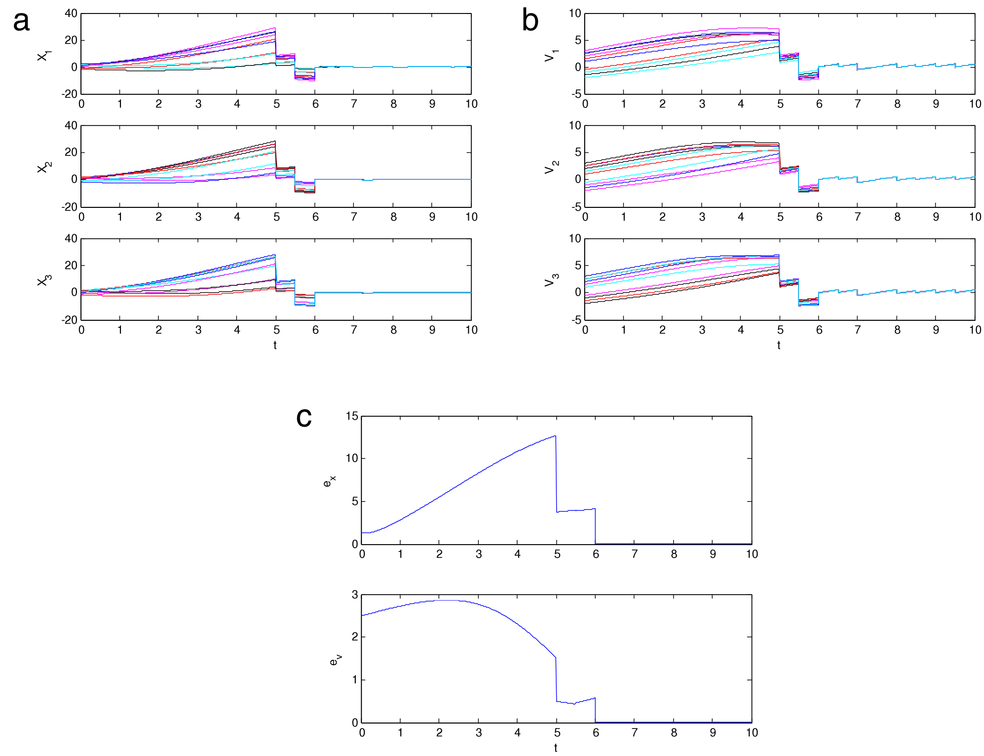

Fig. 3. (a) The evolution of $x(t)$ with the impulsive control. (b) The evolution of $v(t)$ with the impulsive control. (c) The evolution of the total error with the impulsive control.

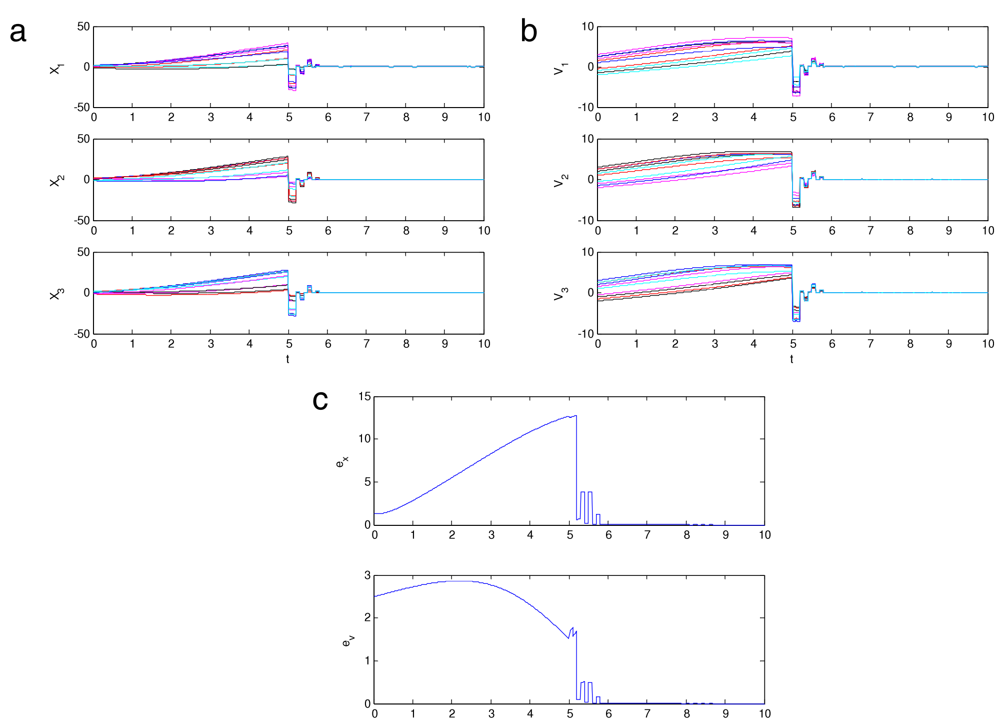

Fig. 4. (a) The evolution of $x(t)$ with the impulsive control. (b) The evolution of $v(t)$ with the impulsive control. (c) The evolution of the total error with the impulsive control.

evolutions of $x(t)$, $v(t)$, and $e(t)$ of the agents are shown in Fig. 4. Note that, by comparing Examples 1 and 2, it can be seen that, as the delay term τ_k increases when τ_k/Δ_k is kept fixed, the norm of B_{A_k} must be small enough in order to offset the influence of τ_k, and only thus the consensus of the agents can be achieved. By comparing the results in Examples 1 and 2, it is concluded that better performance of consensus can be expected when the multi-agent system has smaller delay term τ_k.

6. Conclusion

In this paper, a time-delayed impulsive consensus control scheme has been developed to solve the consensus problem of second-order multi-agent systems with switching topologies. A new comparison principle method was proposed and several effective sufficient conditions were obtained. Compared with the existent works in the literature, the weight matrix of the coupling graph can be negative, which implies that the multi-agent system considered here include both cooperation and competition relationships. Besides, it is not necessary to assume that the interaction topology is strongly connected or contains a directed spanning tree. Finally, numerical simulations validated our theoretical results. In the future, different intrinsic dynamics need to be considered and the corresponding control schemes need to be designed.

Acknowledgments

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