Reverse Group Consensus of Multi-Agent Systems in the Cooperation-Competition Network

Hong-xiang Hu, Wenwu Yu, *Senior Member, IEEE*, Guanghui Wen, *Member, IEEE*, Qi Xuan, and Jinde Cao, *Fellow, IEEE*

***Abstract*—In this paper, a reverse group consensus problem is investigated for the dynamic agents with the inputs in the cooperation-competition network which can be divided into two sub-networks. The weights between the agents in the same sub-network are positive, while the weights between the agents among different sub-networks are negative. Then, the reverse group consensus is firstly studied without the in-degree balance condition. By defining the mirror graph and establishing the solution of the multi-agent system, it is found that the reverse group consensus problem can be achieved if the mirror graph is strongly connected. The explicit expression of the error level is also derived, which would be vanished for multi-agent systems with some special kinds of inputs. Furthermore, as an extension, the decomposing of the cooperation-competition network is discussed, where the concept of the condensation undirected graph and the path balance condition are defined, and several effective sufficient conditions are obtained. Finally, numerical simulation demonstrates the effectiveness of the theoretical analysis.**

***Index Terms*—Condensation undirected graph, cooperation and competition, mirror graph, path balance, reverse group consensus.**

I. Introduction

IN THE last decade, distributed coordination of multi-agent systems has been the subject of much ongoing research, which can not only be used to explain the behavior of natural systems, but also arise in broad areas of applications, including state agreement of the interconnected system [1]–[4], information fusion in wireless sensor networks [5], [6], flocking of mobile agents [7]–[9], formation control of multiple robots [10]–[12], and so on. Please refer to [13], [14] for a more comprehensive overview of the field.

As an important research topic in the distributed coordination of multi-agent systems, group consensus (also called cluster synchronization) has attracted much attention from various scientific communities [15]–[20]. Roughly speaking, the whole network in the group consensus problem will be divided into multiple sub-networks with information communication between them, and its research focuses involve the design of appropriate protocol to drive the agents in the same sub-networks to an agreement.

Encouragingly, great deals of excellent research results about group consensus have emerged constantly. In [15], Yu and Wang considered the group consensus problem in a multi-agent system with switching topologies and communication delays. Furthermore, by introducing double-tree-form transformation, the dynamic equation of agents was transformed into a reduced-order system, and some sufficient conditions were presented. It should be noted that the protocols proposed in [15] are purely continuous-time ones, which means that the channels between different groups must exist continuously. Based on the above framework, Hu *et al.* [16] investigated the group consensus problem with the novel hybrid protocol in case of discontinuous information transmissions among different groups, and some algebraic criteria to guarantee the group consensus were established. In [17], Wang and Cao studied the cluster synchronization problem in nonlinearly coupled non-identical dynamical systems, and obtained some useful synchronization criteria by applying pinning control to a fraction of network agents. Moreover, Su *et al.* [18] considered the pinning control problem for cluster synchronization of undirected complex dynamical networks, and proposed a novel decentralized adaptive pinning-control scheme on both coupling strengths and feedback gains. More recently, Qin and Yu [19] investigated the cluster consensus control for generic linear MASs under fixed and switching directed interaction topology, and it was shown that the cluster consensus behavior was irrelevant to the magnitude of the couplings among agents under the interaction topologies with acyclic partition. Furthermore, based on the above results, Yu *et al.* [20] studied the cluster synchronization for network of linear systems via a generalized pinning control strategy, and it was

Manuscript received May 3, 2016; revised June 28, 2016; accepted July 4, 2016. Date of current version October 25, 2016. This work was supported by the National Natural Science Foundation of China under Grant 61503103, Grant 61572439, Grant 61573096, Grant 61322302, Grant 61304168, Grant 61673107, and Grant 61273212, by the National Ten Thousand Talent Program for Young Top-notch Talents, by Zhejiang Provincial Natural Science Foundation of China under Grant LQ14F030011, by the Natural Science Foundation of Jiangsu Province of China under Grant BK20130595, by the Open Foundation of First Level Zhejiang Key in Key Discipline of Control Science and Engineering, and by the Information Processing and Automation Technology Prior Discipline of Zhejiang Province-Open Research Foundation under Grant 20150709. This paper was recommended by Associate Editor G. Russo.

H. Hu is with the Department of Mathematics, Southeast University, Nanjing 210096, China, and also with the Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, China (e-mail: kukunan911@gmail.com).

W. Yu and J. Cao are with the Department of Mathematics, Southeast University, Nanjing 210096, China (e-mail: wwyu@seu.edu.cn; jdcaoseu@gmail.com).

G. Wen is with the Department of Mathematics, Southeast University, Nanjing 210096, China, and also with the Key Laboratory of Measurement and Control of Complex Systems of Engineering, Ministry of Education, Southeast University, Nanjing 210096, China (e-mail: wenguanghui@gmail.com).

Q. Xuan is with the College of Information Engineering, Zhejiang University of Technology, Hangzhou 310023, China (e-mail: crestxq@hotmail.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSI.2016.2591264

proven that the cluster synchronization can be achieved if the induced network topology of each cluster has a directed spanning tree. In [21], Ji *et al.* elaborated the group consensus of the linearly coupled multi-agent systems including first-order and second-order, respectively, and two novel linear protocols were proposed.

However, almost all the aforementioned results are based on some conservative assumptions, which need to be further generalized in several cases of real applications. For example, in [15]–[20], it is required that the weights between different sub-networks satisfy the in-degree balance condition, i.e., the sum of adjacent weights from every agent in one sub-network to all agents in the other sub-network should be equal to zero, which is too strong to make the multi-agent systems achieve group consensus in reality. Moreover, the in-degree balance condition has been removed in [21], but it is required that the adjacent weights between different sub-networks are non-negative, hence the topology considered in [21] is the cooperation network. Recently, competition relationship between agents has been considered in [22]–[24], which is inevitable due to the limited resources. Thus, it is more practical to investigate the group consensus problem in the cooperation-competition network. It needs to be emphasized that the adjacent weights between different sub-networks rely on the in-degree balance condition is permitted to be negative, which means that the topology in [15]–[20] can be the cooperation-competition network. Meanwhile, the group consensus in [15] and [16] is the static case, where the consensus state of every sub-network is a constant, and it also limits its practical applications. Furthermore, in [17]–[20], Pinning control strategy [25], [26] has been proposed to force the agents in the same sub-network to follow a desired trajectory, which is a dynamic group consensus. Note that in pinning control framework, one need to determine how many agents to pin and to design the pinning control gains. However, it is still a quite difficult problem to find at least how many agents should be pinned for a given network in reality, and the pinning control gains obtained in [17]–[20] are very conservative, which are much larger than the needed values.

Mainly with the above inspirations, in this paper, we focus on a dynamic group consensus problem in the cooperation-competition network without the in-degree balance condition and pinning control strategy. Specifically, in our framework, the whole network is divided into two sub-networks, and the weights between the agents in the same sub-network are positive, while the weights between the agents among different sub-networks are negative. It means that the agents cooperate with their neighbors in the same sub-network, while they compete with the neighbors from different sub-networks. Then, the couple-group consensus problem is firstly studied, and the concept of a mirror graph of a cooperation-competition network is defined to solve this problem. Moreover, it is proven that the couple-group consensus problem can be achieved provided that the mirror graph of the cooperation-competition network is strongly connected. Besides, it is not necessary to assume that the interaction topology of each sub-network is strongly connected or contains a directed spanning tree, instead competition relationship plays a positive role in the dynamic group consensus. Meanwhile, it should be noted that the sum of the couple-group consensus states is zero (called reverse group consensus in this paper), which is sometimes referred as bipartite consensus [27]–[31]. Moreover, bipartite consensus in [28] has been studied by the characteristic of signed graph [29], and it was shown that bipartite consensus can be achieved when the signed graph of the network is structurally balanced. Interestingly, signed graph can be used for predicting the outcome of an opinion forming process in a social network [30]. Recently, interval bipartite consensus has been investigated in [31], which has been proved that all the agents can reach interval bipartite consensus if the associated signed digraph has a spanning tree. Furthermore, as an extension, the decomposing of the cooperation-competition network is discussed. The concept of a condensation undirected graph of a cooperation-competition network, as well as the path balance condition, is defined. It is found that if the condensation undirected graph is path balance, the cooperation-competition network can be divided into two sub-networks with competition relationship between them.

An outline of this paper is as follows. In Section II, some basic definitions in the cooperation-competition network and system model are provided. In Section III, reverse group consensus is proposed, and the main analytical results are obtained. In Section IV, a numerical example is presented to verify the theoretical analysis. Finally, Section V gathers our conclusions and ideas for future work.

Throughout this paper, R and R^n denote the set of real numbers and the set of $n \times 1$ real column vectors, respectively, and $1_n = [1, \ldots, 1]^T \in R^n$. $(\cdot)^T$ denotes transpose, and $\oplus$ denotes the direct sum. The Euclidean norms of a vector $x \in R^n$ and a matrix $A \in R^{n\times n}$ are denoted by $\|x\| = \sqrt{x^T x}$ and $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ with $\lambda_{\max}(\cdot)$ being the maximum eigenvalues of the matrix, respectively. $A = \text{diag}(A_1, \ldots, A_n)$ denotes a block diagonal matrix with the matrices A_i $(i = 1, \ldots, n)$ on the main diagonal, arctg$(\cdot)$ is the arc tangent function. The upper limit of $f(t) : [0, +\infty)$ is denoted by $\overline{\lim}_{t\to\infty} f(t) = \lim_{t\to\infty} \sup_{h\geq t} \{f(h)\}$. Moreover, for $x \in R^n$, span$\{x\}$ and $\{\text{span}\{x\}\}^{\perp}$ denote the spanning space of the vector x and the orthogonal complement of the spanning space of the vector x, respectively.

II. Mathematical Preliminaries

In this section, some basic definitions in the cooperation-competition network and system model are firstly introduced for the subsequent use.

A. Topology Description

In general, information communication between agents in a multi-agent system can be modeled by a network or graph [13], [14]. Specifically, let $\Re = (V, \zeta, A)$ be a weighted directed network with a vertex set $V = \{\pi_1, \pi_2, \ldots, \pi_N\}$, an edge set $\zeta \subseteq V \times V$, and a weighted matrix $A = [a_{ij}]$ representing the communication topology. Moreover, an edge means that there is a directed path form π_j to π_i of $\Re$, and is denoted by $e_{ij} = (\pi_i, \pi_j)$ which is associated with a nonzero weight, i.e., $(\pi_i, \pi_j) \in \zeta \Leftrightarrow a_{ij} \neq 0$, and the neighbor set of agent π_i is denoted by $N_i = \{\pi_j | (\pi_j, \pi_i) \in \zeta\}$. A directed path of length l in $\Re = (V, \zeta, A)$ is a sequence of edges in a directed graph of the form $((\pi_{i_1}, \pi_{i_2}), (\pi_{i_2}, \pi_{i_3}), \ldots, (\pi_{i_l}, \pi_{i_{l+1}}))$, where $(\pi_{i_j}, \pi_{i_{j+1}}) \in \zeta$ for $j = 1, \ldots, l$. Here, it should be noted that $a_{ij} < 0$ also makes sense, which means that agent π_j competes with agent π_i, and such a network is called the cooperation-competition network [22]–[24]. Then, for every agent π_i, the neighbor set can be divided into two parts, i.e., $N_i = \{\pi_j | a_{ij} > 0\} \bigcup \{\pi_j | a_{ij} < 0\}$, which are denoted by N_{i+} and N_{i-} representing the cooperation and competition neighbors of π_i, respectively.

Next, let us introduce a new directed graph $\overline{\Re} = (\overline{V}, \overline{\zeta}, \overline{A})$ with the same vertex set in $\Re$, there is an edge between a pair of distinct agents π_i and π_j if and only if $(\pi_i, \pi_j) \in \zeta(\Re)$, and the weight of the edge (π_i, π_j) in $\overline{\Re}$ is defined as $|a_{ij}|$. The new directed graph $\overline{\Re} = (\overline{V}, \overline{\zeta}, \overline{A})$ is called the mirror graph of the cooperation-competition network $\Re = (V, \zeta, A)$. Note that the mirror graph is the cooperation network due to non-negative weight. Furthermore, to investigate the dynamic group consensus problem, the whole network will be divided into multiple sub-networks, then denote $\Re_c$ as the condensation undirected graph of the cooperation-competition network $\Re$, which is defined as follows: the vertex set consists of all the sub-networks in $\Re$, i.e., denote $\widetilde{\pi}_i$ as sub-network i, and there is an edge between a pair of distinct sub-networks $\widetilde{\pi}_i$ and $\widetilde{\pi}_j$ if and only if there exists $\pi_{i'} \in V(\widetilde{\pi}_i)$ and $\pi_{j'} \in V(\widetilde{\pi}_j)$, such that $(\pi_{i'}, \pi_{j'}) \in \zeta(\Re)$.

B. System Model

Motivated by the work [32], [33], we consider the dynamic group consensus problem in the cooperation-competition network. In our framework, the whole network is divided into multiple sub-networks, and the weights between the agents in the same sub-network are positive, whereas the weights between the agents among different sub-networks are negative. Furthermore, each agent with time-varying input has the following dynamics:

$$\begin{cases} \dot{x}_i(t) = \dot{r}_i(t) - \alpha(x_i - r_i) + \beta \sum\limits_{j \in N_{i+}} a_{ij}(x_j - x_i) \\ \qquad + \gamma \sum\limits_{j \in N_{i-}} a_{ij}(x_j + x_i) + v_i \\ \dot{v}_i(t) = \alpha\beta \sum\limits_{j \in N_{i+}} a_{ij}(x_j - x_i) + \alpha\gamma \sum\limits_{j \in N_{i-}} a_{ij}(x_j + x_i) \end{cases} \tag{1}$$

where $x_i, v_i \in R$ are variables associated with agent i, and $r_i(t)$ is the input of agent i. Obviously, the inputs of the agents varies from one to another, and it makes our results more general. Moreover, α, β, and γ are the positive constants, which can be used to tune the algorithm performance.

Remark 1: In (1), the term $a_{ij}(x_j - x_i)$ represents the information communication between the cooperation neighbor j and agent i, which is the same as those in [13] and [14], and while the term $a_{ij}(x_j + x_i)$ is considered as the information communication between the competition neighbor j and agent i. Moreover, it is easy to see that the competition term is different from those in [22] and [24]. Due to the fact that $a_{ij}(x_j + x_i) = (-a_{ij})(-x_j - x_i)$, competition relationship in this paper is defined as that agent j competes against agent i if and only if the inverse of agent j cooperates with agent i.

III. Main Results

In this section, the reverse group consensus problem for multi-agent system (1) is investigated. The convergence analysis is presented and several effective sufficient conditions are established. Furthermore, the decomposing of the cooperation-competition network as an extension is also discussed.

Firstly, suppose the whole network consists of two sub-networks, and the agents have the dynamics (1). Without loss of generality, let a network topology $\Re = (V, \zeta, A)$ consist of $n + m (n, m > 1)$ agents. Then the first n agents constitute a sub-network $\Re_1$ with the Laplacian matrix L_1, and the rest m agents constitute the other sub-network $\Re_2$ with the Laplacian matrix L_2. Meanwhile, it is assumed that the weights in the same sub-network are positive, while the weights between different sub-networks are negative. Furthermore, denote the index sets $I_1 = \{1, 2, \ldots, n\}$, $I_2 = \{n+1, n+2, \ldots, n+m\}$. In order to obtain the main results, the following definition is needed.

Definition 1: For the cooperation-competition network $\Re$ with two sub-networks, the multi-agent system (1) is said to asymptotically achieve quasi-reverse group consensus under the condition that $(1/n)\sum_{j \in I_1} r_j(t) = -(1/m)\sum_{j \in I_2} r_j(t)$, if for any initial states of systems (1), there exists a parameter $\sigma > 0$, such that the states of agents satisfy $\overline{\lim}_{t \to \infty} \| x_i(t) - (1/n)\sum_{j \in I_1} r_j(t) \| \le \sigma$, $\forall i \in I_1$, and $\overline{\lim}_{t \to \infty} \| x_i(t) - (1/m)\sum_{j \in I_2} r_j(t) \| \le \sigma$, $\forall i \in I_2$. Furthermore, if $\sigma = 0$, the multi-agent system (1) is said to asymptotically achieve reverse group consensus.

Remark 2: The parameter σ in **Definition 1** is called the error level [34]–[36], which reflects the deviation degree of $(1/n)\sum_{j \in I_1} r_j(t)$ (or $(1/m)\sum_{j \in I_2} r_j(t)$). It should be noted that the inputs of the agents are required to satisfy the condition that $(1/n)\sum_{j \in I_1} r_j(t) = -(1/m)\sum_{j \in I_2} r_j(t)$, which is necessary to achieve reverse group consensus for the multi-agent system (1). In fact, when the multi-agent system (1) achieves reverse group consensus, information communication between the agents should vanish, i.e., $\sum_{j \in N_{i-}} a_{ij}((1/n)\sum_{j \in I_1} r_j(t) + (1/m)\sum_{j \in I_2} r_j(t)) = 0$, $\forall i \in I_1 \bigcup I_2$, and one has $(1/n)\sum_{j \in I_1} r_j(t) = -(1/m)\sum_{j \in I_2} r_j(t)$.

Next, the term $\beta\sum_{j \in N_{i+}} a_{ij}(x_j - x_i) + \gamma\sum_{j \in N_{i-}} a_{ij}(x_j + x_i)$ in (1) represents information communication between the agents in the cooperation-competition network, which can be rewritten with a compact form (2), shown at the bottom of the next page, and the matrices in (2) are denoted as follows:

$$\Lambda_1 = \operatorname{diag}\left\{ \sum_{j \in I_2} a_{1j}, \ldots, \sum_{j \in I_2} a_{nj} \right\}$$

$$\Lambda_2 = \operatorname{diag}\left\{ \sum_{j \in I_1} a_{n+1j}, \ldots, \sum_{j \in I_1} a_{n+mj} \right\}$$

$$B_1=\begin{bmatrix} a_{1n+1} & \cdots & a_{1n+m} \\ \vdots & \vdots & \vdots \\ a_{nn+1} & \cdots & a_{nn+m}\end{bmatrix}\in R^{n\times m}$$

$$B_2=\begin{bmatrix} a_{n+11} & \cdots & a_{n+1n} \\ \vdots & \vdots & \vdots \\ a_{n+m1} & \cdots & a_{n+mn}\end{bmatrix}\in R^{m\times n}.$$

Furthermore, denote $\Gamma=\begin{bmatrix}\gamma\Lambda_1-\beta L_1 & \gamma B_1 \\ \gamma B_2 & \gamma\Lambda_2-\beta L_2\end{bmatrix}$, and it is obvious that matrix $\Gamma=[\Gamma_{ij}]$ affects the behavior of the agents in the cooperation-competition network. Hence, before the main results are introduced, the following lemma combines Gershgorin disk theorem to provide spectral characterization of matrix $\Gamma=[\Gamma_{ij}]$ of the cooperation-competition network $\Re$.

Lemma 1: Suppose that matrix $\Gamma=[\Gamma_{ij}]$ represents a cooperation-competition network $\Re$ with two sub-networks, then the following statements hold:

(i) All of the nonzero eigenvalues of $\Gamma=[\Gamma_{ij}]$ have negative real parts;
(ii) Matrix $\Gamma=[\Gamma_{ij}]$ has a zero eigenvalue with the eigenvector $\left[1_n^T,-1_m^T\right]^T\in R^{n+m}$;
(iii) If the mirror graph $\overline{\Re}$ of the cooperation-competition network $\Re$ is strongly connected, then zero is a simple eigenvalue of $\Gamma=[\Gamma_{ij}]$.

Proof: According to the definition of matrix $\Gamma=[\Gamma_{ij}]$, one can obtain that

$$\Gamma_{ii}=\begin{cases}\gamma\sum\limits_{j\in I_2}a_{ij}-\beta\sum\limits_{j\in I_1}a_{ij} & \forall\, i\in I_1\\ \gamma\sum\limits_{j\in I_1}a_{ij}-\beta\sum\limits_{j\in I_2}a_{ij} & \forall\, i\in I_2\end{cases}\tag{3}$$

and one has $\Gamma_{ii}\le 0,\forall\, i\in I_1\bigcup I_2$. Moreover, it follows that:

$$\sum_{j\ne i}|\Gamma_{ij}|=\begin{cases}-\gamma\sum\limits_{j\in I_2}a_{ij}+\beta\sum\limits_{j\in I_1}a_{ij} & \forall\, i\in I_1\\ -\gamma\sum\limits_{j\in I_1}a_{ij}+\beta\sum\limits_{j\in I_2}a_{ij} & \forall\, i\in I_2.\end{cases}\tag{4}$$

Hence, one concludes that $\Gamma_{ii}+\sum_{j\ne i}|\Gamma_{ij}|=0,\forall\, i\in I_1\bigcup I_2$, and Gershgorin disk theorem implies that all of the nonzero eigenvalues of $\Gamma=[\Gamma_{ij}]$ have negative real parts, and this completes the proof of (i). Furthermore, one can observe that

$$\begin{aligned}\Gamma\cdot\begin{bmatrix}1_n\\-1_m\end{bmatrix}&=\begin{bmatrix}\gamma\Lambda_1-\beta L_1 & \gamma B_1\\ \gamma B_2 & \gamma\Lambda_2-\beta L_2\end{bmatrix}\cdot\begin{bmatrix}1_n\\-1_m\end{bmatrix}\\ &=\begin{bmatrix}\gamma\Lambda_1 1_n-\gamma B_1 1_m\\ \gamma B_2 1_n-\gamma\Lambda_2 1_m\end{bmatrix}=0\end{aligned}\tag{5}$$

and it follows that Matrix $\Gamma=[\Gamma_{ij}]$ has a zero eigenvalue with the eigenvector $[1_n^T,-1_m^T]^T\in R^{n+m}$, which completes the proof of (ii). Finally, to proof (iii), denote an auxiliary variable $Z=[z_1,\ldots,z_n,z_{n+1},\ldots,z_{n+m}]^T$, and an auxiliary system is constructed as follows:

$$\begin{cases}\dot z_1=\beta\sum\limits_{j\in N_{1+}}a_{1j}(z_j-z_1)+\gamma\sum\limits_{j\in N_{1-}}a_{1j}(z_j+z_1)\\ \quad\vdots\\ \dot z_n=\beta\sum\limits_{j\in N_{n+}}a_{nj}(z_j-z_n)+\gamma\sum\limits_{j\in N_{n-}}a_{nj}(z_j+z_n)\\ \dot z_{n+1}=\beta\sum\limits_{j\in N_{n+1+}}a_{n+1j}(z_j-z_{n+1})\\ \qquad+\gamma\sum\limits_{j\in N_{n+1-}}a_{n+1j}(z_j+z_{n+1})\\ \quad\vdots\\ \dot z_{n+m}=\beta\sum\limits_{j\in N_{n+m+}}a_{n+mj}(z_j-z_{n+m})\\ \qquad+\gamma\sum\limits_{j\in N_{n+m-}}a_{n+mj}(z_j+z_{n+m})\end{cases}\tag{6}$$

i.e., $\dot Z(t)=\Gamma Z(t)$. Furthermore, denote $y_1=z_1,\ldots,y_n=z_n$, $y_{n+1}=-z_{n+1},\ldots,y_{n+m}=-z_{n+m}$, and $Y=[y_1,\ldots,y_n,y_{n+1},\ldots,y_{n+m}]^T$, then one has

$$\begin{cases}\dot y_1=\beta\sum\limits_{j\in N_{1+}}a_{1j}(y_j-y_1)+\gamma\sum\limits_{j\in N_{1-}}(-a_{1j})(y_j-y_1)\\ \quad\vdots\\ \dot y_n=\beta\sum\limits_{j\in N_{n+}}a_{nj}(y_j-y_n)+\gamma\sum\limits_{j\in N_{n-}}(-a_{nj})(y_j-y_n)\\ \dot y_{n+1}=\beta\sum\limits_{j\in N_{n+1+}}a_{n+1j}(y_j-y_{n+1})\\ \qquad+\gamma\sum\limits_{j\in N_{n+1-}}(-a_{n+1j})(y_j-y_{n+1})\\ \quad\vdots\\ \dot y_{n+m}=\beta\sum\limits_{j\in N_{n+m+}}a_{n+mj}(y_j-y_{n+m})\\ \qquad+\gamma\sum\limits_{j\in N_{n+m-}}(-a_{n+mj})(y_j-y_{n+m}).\end{cases}\tag{7}$$

$$\begin{bmatrix}\beta\sum\limits_{j\in N_{1+}}a_{ij}(x_j-x_1)+\gamma\sum\limits_{j\in N_{1-}}a_{ij}(x_j+x_1)\\ \vdots\\ \beta\sum\limits_{j\in N_{n+}}a_{ij}(x_j-x_n)+\gamma\sum\limits_{j\in N_{n-}}a_{ij}(x_j+x_n)\\ \beta\sum\limits_{j\in N_{n+1+}}a_{ij}(x_j-x_{n+m})+\gamma\sum\limits_{j\in N_{n+1-}}a_{ij}(x_j+x_{n+m})\\ \vdots\\ \beta\sum\limits_{j\in N_{n+m+}}a_{ij}(x_j-x_{n+m})+\gamma\sum\limits_{j\in N_{n+m-}}a_{ij}(x_j+x_{n+m})\end{bmatrix}=\begin{bmatrix}\gamma\Lambda_1-\beta L_1 & \gamma B_1\\ \gamma B_2 & \gamma\Lambda_2-\beta L_2\end{bmatrix}\cdot\begin{bmatrix}x_1\\ \vdots\\ x_n\\ \vdots\\ x_{n+m}\end{bmatrix}\tag{2}$$

i.e., $\dot{Y}(t)=LY(t)$, where $L=\begin{bmatrix}-\gamma\Lambda_1-\beta L_1 & -\gamma B_1\\ -\gamma B_2 & -\gamma\Lambda_2-\beta L_2\end{bmatrix}$. Moreover, it should be noted that the network with the Laplacian matrix L has the same set of edges with the mirror graph $\overline{\Re}$. Due to the fact that the mirror graph $\overline{\Re}$ of the cooperation-competition network $\Re$ is strongly connected, one can conclude that zero is a simple eigenvalue of L. Hence, for (7) with any initial state $Y(0)$, there exists a constant η, such that $\lim_{t\to\infty} Y(t)=\eta 1_{n+m}$, and it follows that for (6) with any initial state $Z(0)$, there exists a constant η, such that $\lim_{t\to\infty} Z(t)=\eta\left[1_n^T,-1_m^T\right]^T$, which implies that zero is a simple eigenvalue of $\Gamma=[\Gamma_{ij}]$. This completes the proof.

Remark 3: In **Lemma 1**, the spectral characterization of matrix Γ is provided by constructing the auxiliary system (6). Actually, with the variable replacement $Y=\begin{bmatrix}I_n & 0\\ 0 & -I_m\end{bmatrix}Z$, the cooperation-competition network $\Re$ can be transformed into the cooperative network, i.e., the mirror graph $\overline{\Re}$. Here, it is required that the mirror graph should be strongly connected, which is essentially structurally balanced condition [27]–[30] in the interaction topology of each agent.

Then, the main result of the paper is given by the following theorem:

Theorem 1: Consider a cooperation-competition network $\Re$ with $n+m$ agents governed by the form (1), and assume that the mirror graph $\overline{\Re}$ of the cooperation-competition network $\Re$ is strongly connected. If the inputs of the agents in the same sub-network satisfies $\lim_{t\to\infty}(r_i(t)-r_j(t))=0$, then for any initial states $x_i(0)$ and $v_i(0)$ with $\sum_{i\in I_1} v_i(0)=\sum_{i\in I_2} v_i(0)$, the multi-agent system (1) asymptotically achieve quasi-reverse group consensus with the error level $\sigma=c\|\Gamma\|/\rho$, where ρ is the minimal positive real part of the eigenvalues of matrix $-\Gamma$, and c is a constant depending on the inputs of the agents. Furthermore, if there exist δ and $\xi>0$ such that the inputs of the agents in the same sub-network satisfies $\lim_{t\to\infty} e^{\xi t}(r_i(t)-r_j(t))=\delta$, the multi-agent system (1) asymptotically achieves reverse group consensus.

Proof: Firstly, denote the auxiliary variables $X_1=[x_1,\ldots,x_n]^T$, $X_2=[x_{n+1},\ldots,x_{n+m}]^T$, $V_1=[v_1,\ldots,v_n]^T$, $V_2=[v_{n+1},\ldots,v_{n+m}]^T$, $r^{(1)}=[r_1,\ldots,r_n]^T$, $r^{(2)}=[r_{n+1},\ldots,r_{n+m}]^T$, $\bar{r}^{(1)}=(1/n)\sum_{j=1}^n r_j\cdot 1_n$, $\bar{r}^{(2)}=(1/m)\sum_{j=n+1}^{n+m} r_j\cdot 1_m$, $Y_1=X_1-\bar{r}^{(1)}$, and $Y_2=X_2-\bar{r}^{(2)}$, then according to (1), one has (8), shown at the bottom of the page, and $Q_n=I_n-(1/n)1_n\cdot 1_n^T$. Let $W_1=V_1+\alpha Q_n r^{(1)}$, (8) can be rewritten as

$$\dot{Y}_1=-\alpha Y_1+Q_n\dot{r}^{(1)}-\beta L_1X_1+\gamma\Lambda_1X_1+\gamma B_1X_2+W_1. \tag{9}$$

Moreover, for the variable W_1, one can obtain that

$$\begin{aligned}\dot{W}_1&=\dot{V}_1+\alpha Q_n\dot{r}^{(1)}\\&=-\alpha\beta L_1X_1+\alpha\gamma\Lambda_1X_1+\alpha\gamma B_1X_2+\alpha Q_n\dot{r}^{(1)}.\end{aligned} \tag{10}$$

Then, according to (9) and (10), it follows that:

$$\dot{Y}_1=-\alpha Y_1+W_1+\frac{\dot{W}_1}{\alpha} \tag{11}$$

i.e., $[Y_1-(W_1/\alpha)]'=-\alpha[Y_1-(W_1/\alpha)]$, thus the solution of (11) is $Y_1(t)-(W_1(t)/\alpha)=e^{-\alpha t}[Y_1(0)-(W_1(0)/\alpha)]$, which implies that $\lim_{t\to\infty}[Y_1(t)-(W_1(t)/\alpha)]=0$. Similar to (9) and (10), one has

$$\begin{aligned}\dot{Y}_2=&-\alpha Y_2+Q_m\dot{r}^{(2)}-\beta L_2X_2\\&+\gamma\Lambda_2X_2+\gamma B_2X_1+W_2\end{aligned} \tag{12}$$

$$\dot{W}_2=-\alpha\beta L_2X_2+\alpha\gamma\Lambda_2X_2+\alpha\gamma B_2X_1+\alpha Q_m\dot{r}^{(2)} \tag{13}$$

where $Q_m=I_m-(1/m)1_m\cdot 1_m^T$, and $W_2=V_2+\alpha Q_m r^{(2)}$. Furthermore, it follows from (12) and (13) that $Y_2(t)-(W_2(t)/\alpha)=e^{-\alpha t}[Y_2(0)-(W_2(0)/\alpha)]$, and $\lim_{t\to\infty}[Y_2(t)-(W_2(t)/\alpha)]=0$. Denote $\eta_1=Y_1(0)-(W_1(0)/\alpha)$, and $\eta_2=Y_2(0)-(W_2(0)/\alpha)$, then one obtains $Y_1(t)=(W_1(t)/\alpha)+e^{-\alpha t}\eta_1$ and $Y_2(t)=(W_2(t)/\alpha)+e^{-\alpha t}\eta_2$. From (10), one can deduce that

$$\begin{aligned}\dot{W}_1&=-\alpha\beta L_1X_1+\alpha\gamma\Lambda_1X_1+\alpha\gamma B_1X_2+\alpha Q_n\dot{r}^{(1)}\\&=-\alpha\beta L_1\left(Y_1+\bar{r}^{(1)}\right)+\alpha\gamma\Lambda_1\left(Y_1+\bar{r}^{(1)}\right)\\&\quad+\alpha\gamma B_1\left(Y_2+\bar{r}^{(2)}\right)+\alpha Q_n\dot{r}^{(1)}\\&=-\alpha\beta L_1\left(\frac{W_1(t)}{\alpha}+e^{-\alpha t}\eta_1+\bar{r}^{(1)}\right)\\&\quad+\alpha\gamma\Lambda_1\left(\frac{W_1(t)}{\alpha}+e^{-\alpha t}\eta_1+\bar{r}^{(1)}\right)\\&\quad+\alpha\gamma B_1\left(\frac{W_2(t)}{\alpha}+e^{-\alpha t}\eta_2+\bar{r}^{(2)}\right)+\alpha Q_n\dot{r}^{(1)}\\&=(\gamma\Lambda_1-\beta L_1)W_1(t)+\gamma B_1W_2(t)\\&\quad+\alpha e^{-\alpha t}(-\beta L_1\eta_1+\gamma\Lambda_1\eta_1+\gamma B_1\eta_2)\\&\quad+\alpha\gamma\Lambda_1\bar{r}^{(1)}+\alpha\gamma B_1\bar{r}^{(2)}+\alpha Q_n\dot{r}^{(1)}.\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{Y}_1&=\dot{X}_1-\dot{\bar{r}}^{(1)}=\dot{r}^{(1)}-\alpha\left(X_1-r^{(1)}\right)-\beta L_1X_1+\gamma\Lambda_1X_1+\gamma B_1X_2+V_1-\dot{\bar{r}}^{(1)}\\&=\dot{r}^{(1)}-\dot{\bar{r}}^{(1)}-\alpha\left(X_1-\bar{r}^{(1)}+\bar{r}^{(1)}-r^{(1)}\right)-\beta L_1X_1+\gamma\Lambda_1X_1+\gamma B_1X_2+V_1\\&=-\alpha Y_1+\alpha Q_nr^{(1)}+Q_n\dot{r}^{(1)}-\beta L_1X_1+\gamma\Lambda_1X_1+\gamma B_1X_2+V_1\end{aligned} \tag{8}$$

Due to the fact that $(1/n)\sum_{j\in I_1} r_j(t) = -(1/m)\sum_{j\in I_2} r_j(t)$, one has $\alpha\gamma\Lambda_1\bar{r}^{(1)}+\alpha\gamma B_1\bar{r}^{(2)}=0$, and (14) can be rewritten as

$$\dot{W}_1 = \alpha e^{-\alpha t}\left(-\beta L_1\eta_1 + \gamma\Lambda_1\eta_1 + \gamma B_1\eta_2\right) + \alpha Q_n\dot{r}^{(1)} + \left(\gamma\Lambda_1 - \beta L_1\right)W_1(t) + \gamma B_1 W_2(t). \quad (15)$$

Similar to (15), one also has

$$\dot{W}_2 = \alpha e^{-\alpha t}\left(-\beta L_2\eta_2 + \gamma\Lambda_2\eta_2 + \gamma B_2\eta_1\right) + \alpha Q_m\dot{r}^{(2)} + \left(\gamma\Lambda_2 - \beta L_2\right)W_2(t) + \gamma B_2 W_1(t). \quad (16)$$

Moreover, one can use vector notations to rewrite (15) and (16) with a compact form as follows:

$$\begin{bmatrix}\dot{W}_1\\ \dot{W}_2\end{bmatrix} = \Gamma\begin{bmatrix}W_1\\ W_2\end{bmatrix} + \alpha e^{-\alpha t}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} + \alpha\begin{bmatrix}Q_n\dot{r}^{(1)}\\ Q_m\dot{r}^{(2)}\end{bmatrix}. \quad (17)$$

Then, the solution of (17) is

$$\begin{bmatrix}W_1(t)\\ W_2(t)\end{bmatrix} = \alpha\int_0^t e^{\Gamma(t-\tau)}\left\{e^{-\alpha\tau}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} + \begin{bmatrix}Q_n\dot{r}^{(1)}\\ Q_m\dot{r}^{(2)}\end{bmatrix}\right\}d\tau + e^{\Gamma t}\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix} \quad (18)$$

which implies that

$$\left\|\begin{bmatrix}W_1(t)\\ W_2(t)\end{bmatrix}\right\| \le \left\|e^{\Gamma t}\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix}\right\| + \left\|\int_0^t e^{\Gamma(t-\tau)}\alpha e^{-\alpha\tau}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}d\tau\right\| + \left\|\int_0^t e^{\Gamma(t-\tau)}\alpha\begin{bmatrix}Q_n\dot{r}^{(1)}(\tau)\\ Q_m\dot{r}^{(2)}(\tau)\end{bmatrix}d\tau\right\|. \quad (19)$$

Consider the first term in (19), one has

$$\left\|e^{\Gamma t}\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix}\right\| \le e^{-\rho t}\left\|\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix}\right\| \quad (20)$$

where where $\rho > 0$ is the minimal positive real part of the eigenvalues of matrix $-\Gamma$. Specifically, due to the fact that the mirror graph $\overline{\Re}$ of the cooperation-competition network $\Re$ is strongly connected, it follows from **Lemma 1** that zero is a simple eigenvalue of Γ with the eigenvector $[1_n^T, -1_m^T]^T$, and all of the nonzero eigenvalues have negative real parts. Furthermore, given $v_i(0)$ with $\sum_{i\in I_1} v_i(0) = \sum_{i\in I_2} v_i(0)$, and using $1_n^T Q_n = 1_n^T\left[I_n - (1/n)1_n\cdot 1_n^T\right] = 0$ and $1_m^T Q_m = 0$, one can deduce that

$$\left[1_n^T, -1_m^T\right]\cdot\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix} = 1_n^T V_1(0) - 1_m^T V_2(0) + \alpha 1_n^T Q_n r^{(1)}(0) - \alpha 1_m^T r^{(2)}(0) = 0 \quad (21)$$

which implies that $\begin{bmatrix}W_1(0)\\ W_2(0)\end{bmatrix} \in \left\{\text{span}\left\{\left[1_n^T, -1_m^T\right]^T\right\}\right\}^{\perp}$. Thus, one obtains (20).

Next, consider the term $\left\|\int_0^t e^{\Gamma(t-\tau)}\alpha e^{-\alpha\tau}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}d\tau\right\|$, since $R^{n+m} = \text{span}\left\{\left[1_n^T, -1_m^T\right]^T\right\}\oplus\left\{\text{span}\left\{\left[1_n^T, -1_m^T\right]^T\right\}\right\}^{\perp}$, there exist vectors $\begin{bmatrix}\widehat{\eta}_1\\ \widehat{\eta}_2\end{bmatrix} \in \text{span}\left\{\left[1_n^T, -1_m^T\right]^T\right\}$ and $\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix} \in \left\{\text{span}\left\{\left[1_n^T, -1_m^T\right]^T\right\}\right\}^{\perp}$. Note that $\Gamma\begin{bmatrix}\widehat{\eta}_1\\ \widehat{\eta}_2\end{bmatrix} = 0$. Then the term $\left\|\int_0^t e^{\Gamma(t-\tau)}\alpha e^{-\alpha\tau}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}d\tau\right\|$ can be rewritten as

$$\begin{aligned}
&\left\|\int_0^t e^{\Gamma(t-\tau)}\alpha e^{-\alpha\tau}\Gamma\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}d\tau\right\| \\
&= \left\|\int_0^t e^{\Gamma(t-\tau)}\alpha e^{-\alpha\tau}\Gamma\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix}d\tau\right\| \\
&\le \int_0^t \alpha e^{-\alpha\tau}\left\|\Gamma\right\|\cdot\left\|e^{\Gamma(t-\tau)}\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix}\right\|d\tau \\
&\le \int_0^t \alpha e^{-\alpha\tau}\cdot e^{-\rho(t-\tau)}\left\|\Gamma\right\|\cdot\left\|\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix}\right\|d\tau \\
&= \begin{cases}\dfrac{\alpha t}{e^{\alpha t}}\left\|\Gamma\right\|\left\|\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix}\right\|, & \text{if } \alpha = \rho\\[2ex] \dfrac{\alpha(e^{-\alpha t} - e^{-\rho t})}{\rho - \alpha}\left\|\Gamma\right\|\left\|\begin{bmatrix}\breve{\eta}_1\\ \breve{\eta}_2\end{bmatrix}\right\|, & \text{if } \alpha \ne \rho.\end{cases}
\end{aligned} \quad (22)$$

Moreover, for the term $\left\|\int_0^t e^{\Gamma(t-\tau)}\alpha\begin{bmatrix}Q_n\dot{r}^{(1)}(\tau)\\ Q_m\dot{r}^{(2)}(\tau)\end{bmatrix}d\tau\right\|$, it is calculated by

$$\begin{aligned}
&\int_0^t e^{\Gamma(t-\tau)}\alpha\begin{bmatrix}Q_n\dot{r}^{(1)}(\tau)\\ Q_m\dot{r}^{(2)}(\tau)\end{bmatrix}d\tau \\
&= \alpha e^{\Gamma t}\int_0^t e^{-\Gamma\tau}d\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix} \\
&= \alpha e^{\Gamma t}\left\{e^{-\Gamma\tau}\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}\Bigg|_0^t\right\} + \alpha e^{\Gamma t}\left\{\int_0^t e^{-\Gamma\tau}\Gamma\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}d\tau\right\} \\
&= \alpha\begin{bmatrix}Q_n r^{(1)}(t)\\ Q_m r^{(2)}(t)\end{bmatrix} - \alpha e^{\Gamma t}\begin{bmatrix}Q_n r^{(1)}(0)\\ Q_m r^{(2)}(0)\end{bmatrix} + \alpha\int_0^t e^{\Gamma(t-\tau)}\Gamma\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}d\tau.
\end{aligned} \quad (23)$$

Because the inputs of the agents in the same sub-network satisfies $\lim_{t\to\infty}(r_i(t)-r_j(t))=0$, one has $\lim_{t\to\infty}\alpha\begin{bmatrix}Q_n r^{(1)}(t)\\ Q_m r^{(2)}(t)\end{bmatrix}=0$, then the vector $\begin{bmatrix}Q_n r^{(1)}(t)\\ Q_m r^{(2)}(t)\end{bmatrix}$ is bounded, i.e., there exists $c>0$ such that $\left\|\begin{bmatrix}Q_n r^{(1)}(t)\\ Q_m r^{(2)}(t)\end{bmatrix}\right\|\le c$ for any $t\in[0,\infty)$. Furthermore, it follows from $\left[1_n^T,-1_m^T\right]\begin{bmatrix}Q_n r^{(1)}(0)\\ Q_m r^{(2)}(0)\end{bmatrix}=0$ that $\left\|\alpha e^{\Gamma t}\begin{bmatrix}Q_n r^{(1)}(0)\\ Q_m r^{(2)}(0)\end{bmatrix}\right\|\le\alpha e^{-\rho t}\left\|\begin{bmatrix}Q_n r^{(1)}(0)\\ Q_m r^{(2)}(0)\end{bmatrix}\right\|$.

Consider the term $\alpha\int_0^t e^{\Gamma(t-\tau)}\Gamma\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}d\tau$, one can deduce

$$\begin{aligned}&\left\|\alpha\int_0^t e^{\Gamma(t-\tau)}\Gamma\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}d\tau\right\|\\&\le\alpha\|\Gamma\|\int_0^t\left\|e^{\Gamma(t-\tau)}\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}\right\|d\tau\\&\le\alpha\|\Gamma\|\int_0^t e^{-\rho(t-\tau)}\left\|\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}\right\|d\tau\\&\le\alpha c\|\Gamma\|\frac{1-e^{-\rho t}}{\rho}.\end{aligned}\tag{24}$$

According to (20), (22)–(24), one has $\overline{\lim}_{t\to\infty}\left\|\begin{bmatrix}W_1(t)\\ W_2(t)\end{bmatrix}\right\|\le\alpha c\|\Gamma\|\rho$. Then

$$\begin{aligned}&\overline{\lim_{t\to\infty}}\|Y_1(t)\|\\&\le\lim_{t\to\infty}\left\|Y_1(t)-\frac{W_1(t)}{\alpha}\right\|+\overline{\lim_{t\to\infty}}\left\|\frac{W_1(t)}{\alpha}\right\|\\&\le\frac{c\|\Gamma\|}{\rho}.\end{aligned}\tag{25}$$

Similar to (25), one has $\overline{\lim}_{t\to\infty}\|Y_2(t)\|\le c\|\Gamma\|/\rho$, and it means that the multi-agent system (1) asymptotically achieves quasi-reverse group consensus with the parameter $\sigma=c\|\Gamma\|/\rho$.

Furthermore, when the inputs of the agents in the same sub-network satisfies $\lim_{t\to\infty}e^{\xi t}(r_i(t)-r_j(t))=\delta$, there exists $M>0$, such that $\left\|\begin{bmatrix}Q_n r^{(1)}(t)\\ Q_m r^{(2)}(t)\end{bmatrix}\right\|\le Me^{-\xi t}$, and (24) can be rewritten as

$$\begin{aligned}&\left\|\alpha\int_0^t e^{\Gamma(t-\tau)}\Gamma\begin{bmatrix}Q_n r^{(1)}(\tau)\\ Q_m r^{(2)}(\tau)\end{bmatrix}d\tau\right\|\\&\le\alpha c\|\Gamma\|\int_0^t e^{-\rho(t-\tau)}Me^{-\xi\tau}d\tau\\&=\begin{cases}\alpha cM\|\Gamma\|\frac{t}{e^{\rho t}}, & \text{if }\rho=\xi\\ \alpha cM\|\Gamma\|\frac{e^{-\xi t}-e^{-\rho t}}{\rho-\xi}, & \text{if }\rho\ne\xi.\end{cases}\end{aligned}\tag{26}$$

Then, according to (20), (22), (23), and (26), one has $\lim_{t\to\infty}\left\|\begin{bmatrix}W_1(t)\\ W_2(t)\end{bmatrix}\right\|=0$, which implies that $\lim_{t\to\infty}Y_1(t)=\lim_{t\to\infty}Y_2(t)=0$, thus the multi-agent system (1) asymptotically achieves reverse group consensus. This completes the proof.

Remark 4: In **Theorem 1**, by using the mirror graph, the reverse group consensus of the multi-agent system (1) is solved asymptotically. In our framework, the initial states of the agents should satisfy the condition $\sum_{i\in I_1}v_i(0)=\sum_{i\in I_2}v_i(0)$, which yields (20) and (24), then the states of the agents in the same sub-network track the dynamic average of their inputs, and the sum of group consensus states of two sub-networks is zero (called reverse group consensus).

Remark 5: It should be noted that different from [15]–[20], the in-degree balance condition is removed in this paper, and the extra competition relationships between the agents in different sub-networks is considered, where the adjacent weights are negative. Moreover, we do not assume that the sub-network topology is strongly connected or contains a directed spanning tree in Theorem 1, thus the whole network described by matrix $\Gamma=[\Gamma_{ij}]$ has a more general network topology than those considered in the literature [15]–[21], which makes our results more promising in a broad range of practical applications. Meanwhile, competition relationship plays a positive role in the reverse group consensus of this paper. In fact, competition relationship indirectly increases the chance of information communication between the agents in the same sub-network without information communication.

Remark 6: The convergence rate is crucial for the reverse group consensus, which indicates how quickly the agents in the cooperation-competition network achieve it. According to (11), (20), (22), (23), and (26), one can conclude that the multi-agent system (1) achieves reverse couple-group consensus exponentially with the convergence rate $\min\{\alpha,\rho,\xi\}$. Obviously, the convergence rate of reverse group consensus depends on the design parameters α, β, γ, the communication topology represents by matrix $\Gamma=[\Gamma_{ij}]$, and the characteristic of the inputs of the agents. Furthermore, when the multi-agent system (1) achieves quasi-reverse group consensus, the explicit expression of the error level in **Theorem 1** is derived by $\sigma=c\|\Gamma\|/\rho$, which can also tune the parameters α, β, γ to improve the performance of the error level.

Obviously, one precondition in **Theorem 1** is that a cooperation-competition network can be divided into two sub-networks, and the relationship between them is competitive. However, in the practical multi-agent systems, the number of agents may be large, making it quite difficult to verify this premise directly. Therefore, it requires us to seek an effective technical way to solve this problem.

Given a cooperation-competition network $\Re=(V,\zeta,A)$, one may relegate the agents with positive weights to the same sub-network firstly. It should be noted that the number of sub-networks may be larger than two. Without loss of generality, suppose that the cooperation-competition network $\Re$ is composed of $k(k>2)$ sub-networks, and the qth sub-network has n_q agents with the Laplacian matrix L_q, $q=1,2,\ldots,k$. Then, the weights in the same sub-network are positive, and it implies that the agents cooperate with its neighbors in the same sub-network, meanwhile the weights between the agents among different sub-networks are negative, which leads to the fact that the agents compete with its neighbors from different sub-networks. Denote the index sets $I_1 = \{1, 2, \ldots, n_1\}$, $I_2 = \{n_1+1, n_1+2, \ldots, n_1+n_2\}, \ldots, I_k = \{\sum_{i=1}^{k-1} n_i + 1, \sum_{i=1}^{k-1} n_i + 2, \ldots, \sum_{i=1}^{k} n_i\}$, then the path balance condition in the condensation undirected graph $\Re_c$ of the cooperation-competition network $\Re$ is introduced as follows.

Definition 2: The condensation undirected graph $\Re_c$ is path balance if the lengths of all the paths between $\widetilde{\pi}_i$ and $\widetilde{\pi}_j$ have the same parity for each pair of distinct vertexes $(\widetilde{\pi}_i, \widetilde{\pi}_j)$.

Then, the following theorem provides some sufficient condition about the decomposing of the cooperation-competition network.

Theorem 2: Consider a cooperation-competition network $\Re$ is composed of $k (k > 2)$ sub-networks, and assume that the condensation undirected graph $\Re_c$ is path balance, then the cooperation-competition network can be finally divided into two sub-networks with competition relationship between them.

Proof: We first denote the indicated number C_q of the qth sub-network ($\tilde{\pi}_q$ in $\Re_c$), where $C_q \in \{0, 1\}$, $q = 1, 2, \ldots, k$. Then, select a sub-network randomly, and assume that the jth sub-network is selected. Here, the indicated number C_j is defined as 1, i.e., $C_j = 1$. Next, consider the neighbors of the jth sub-network, and the indicated numbers of the neighbors are defined as 0, i.e., $C_i = 0$, $\forall \tilde{\pi}_i \in N_{\tilde{\pi}_j}$, where $N_{\tilde{\pi}_j}$ is the neighbor set of the jth sub-network. Furthermore, for the ith sub-network, the indicated numbers of the neighbors of the ith sub-network are defined as 1, i.e., $C_p = 1$, $\forall \tilde{\pi}_p \in N_{\tilde{\pi}_i}, \tilde{\pi}_i \in N_{\tilde{\pi}_j}$. Note that this process is well-defined. In fact, suppose that this process is not well-defined, then there exist i_0 and p_0, such that $\tilde{\pi}_{i_0} \in N_{\tilde{\pi}_j}, \tilde{\pi}_{p_0} \in N_{\tilde{\pi}_{i_0}}$, and C_{p_0} can be 1 as well as 0. When $C_{p_0} = 0$, one can conclusion that $\tilde{\pi}_{p_0} \in N_{\tilde{\pi}_j}$, there exists a path from $\tilde{\pi}_j$ to $\tilde{\pi}_{p_0}$, i.e., $(\tilde{\pi}_j, \tilde{\pi}_{p_0})$. Meanwhile, if $C_{p_0} = 1$, there also exists a path from $\tilde{\pi}_j$ to $\tilde{\pi}_{p_0}$, i.e., $(\tilde{\pi}_j, \tilde{\pi}_{i_0})$ and $(\tilde{\pi}_{i_0}, \tilde{\pi}_{p_0})$. Obviously, the condensation undirected graph $\Re_c$ is not path balance, and this results in a contradiction. Therefore, by induction we can define all the indicated numbers of the k sub-networks, which indicates that the cooperation-competition network can be finally divided into two sub-networks with competition relationship between them. This completes the proof.

According to **Theorem 2**, when the condensation undirected graph $\Re_c$ is path balance, the vertex set of $\Re_c$ can be divided into two categories, denoted by $\widehat{k}$ and $\breve{k}$, respectively. Next, in this case, information communication between the agents in the whole network is described by matrix $\Gamma = [\Gamma_{ij}]$, and it has the form

$$\Gamma = \begin{bmatrix} \gamma\Lambda_1 - \beta L_1 & \gamma B_{12} & \cdots & \gamma B_{1k} \\ \gamma B_{21} & \gamma\Lambda_2 - \beta L_2 & \cdots & \gamma B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma B_{k1} & \gamma B_{k2} & \cdots & \gamma\Lambda_k - \beta L_k \end{bmatrix} \tag{27}$$

where $\Lambda_i = \operatorname{diag}\left\{\sum_{j \notin I_i} a_{(n_1+\cdots+n_{i-1}+p)j}\right\} \in R^{n_i \times n_i}$, $p = 1, 2, \ldots, n_i$, and $B_{ij} \in R^{n_i \times n_j}$, which represents the information communication from the jth sub-network to the ith sub-network. Then, for the sake of completeness, the following result is given, which is a generalization of **Lemma 1**.

Theorem 3: Suppose that matrix $\Gamma = [\Gamma_{ij}]$ represents a cooperation-competition network $\Re$ with k sub-networks, then the following statements hold:

(i) All of the nonzero eigenvalues of $\Gamma = [\Gamma_{ij}]$ have negative real parts;

(ii) If the mirror graph $\overline{\Re}$ of the cooperation-competition network $\Re$ is strongly connected, and the condensation undirected graph $\Re_c$ is path balance, then zero is a simple eigenvalue of $\Gamma = [\Gamma_{ij}]$ with the eigenvector $\left[(-1)^{C_1} 1_{n_1}^T, (-1)^{C_2} 1_{n_2}^T, \ldots, (-1)^{C_k} 1_{n_k}^T\right]^T \in R^{\sum_{i=1}^k n_i}$, where the parameter C_i is defined as $C_i = \begin{cases} 0, & \text{if } i \in \widehat{k} \\ 1, & \text{if } i \in \breve{k} \end{cases}$.

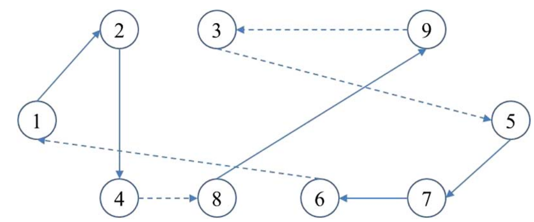

Fig. 1. Interaction topology of nine agents.

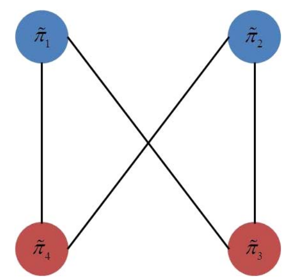

Fig. 2. Condensation undirected graph $\Re_c$.

IV. Numerical Simulation

In this section, a numerical example is provided to illustrate the effectiveness of the proposed theoretical results.

Consider nine agents in the cooperation-competition network, with their interaction topology presented in Fig. 1, and the solid lines represent cooperation relationship between agents, while the dotted lines represent competition relationship between agents.

According to Fig. 1, the mirror graph $\bar{\Re}$ of the cooperation-competition network $\Re$ is strong connected. Then, according to positive weights, one has four sub-networks, i.e., $\Re_1 = \{1, 2, 4\}$, $\Re_2 = \{3\}$, $\Re_3 = \{5, 6, 7\}$, and $\Re_4 = \{8, 9\}$. Next, one obtains the condensation undirected graph $\Re_c$ presented in Fig. 2, which satisfies the path balance condition.

Therefore, from **Theorem 2**, one can conclude that the cooperation-competition network $\Re$ finally can be divided into two sub-networks, i.e., $\Re_1 = \{1, 2, 3, 4\}$, $\Re_2 = \{5, 6, 7, 8, 9\}$, and

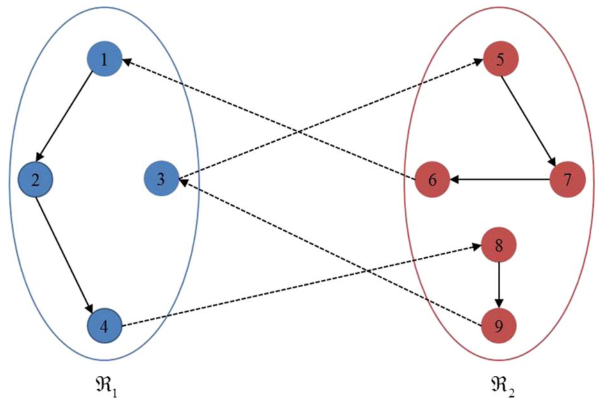

Fig. 3. Two sub-networks in the cooperation-competition network $\Re$.

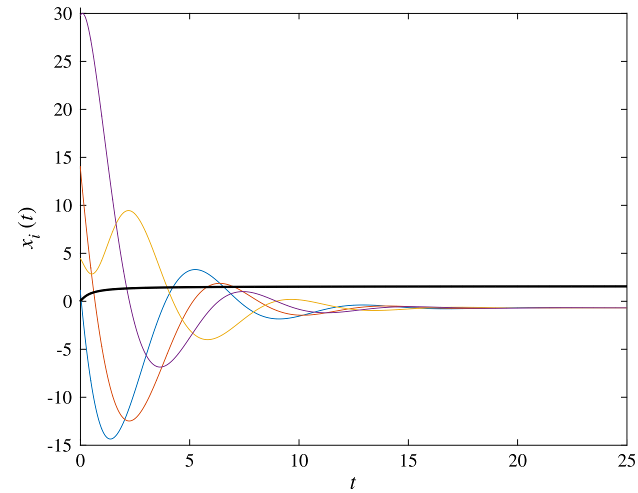

Fig. 4. Evolution of $x_i(t)$ in sub-network $\Re_1$.

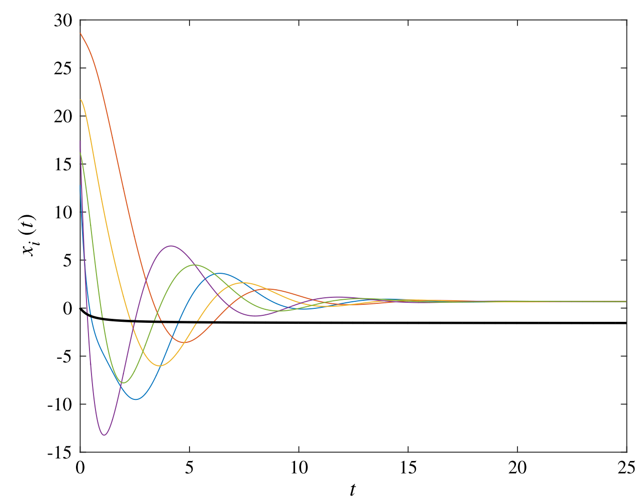

Fig. 5. Evolution of $x_i(t)$ in sub-network $\Re_2$.

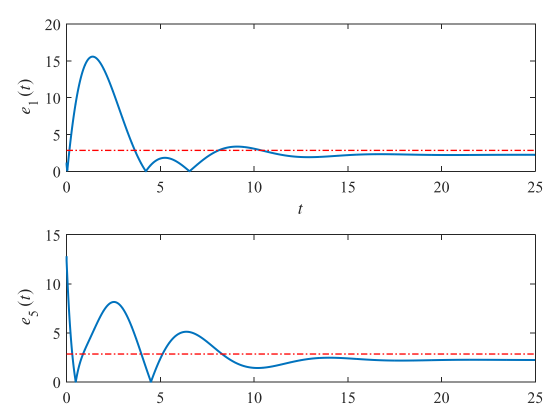

Fig. 6. Evolution of the errors of agent 1 and 5.

the relationship between them is competitive, which is presented in Fig. 3.

Meanwhile, it is satisfied that $a_{ij} = 1$ for $j \in N_{i+}$, and $a_{ij} = -1$ for $j \in N_{i-}$. It should be noted that the topologies of two sub-networks are neither strongly connected nor containing a directed spanning tree. Next, the initial values of $x_i(t)$ are randomly chosen from the interval [0, 30], and the initial values of $v_i(t)$ are randomly chosen from the interval [0, 50], which satisfies the condition in **Theorem 1**, i.e., $\sum_{i=1}^{4} v_i(0) = \sum_{i=5}^{9} v_i(0)$. Moreover, the input of agent i is defined as $r_1(t) = \text{arctg}(t)$, $r_2(t) = \text{arctg}(2t)$, $r_3(t) = \text{arctg}(3t)$, $r_4(t) = \text{arctg}(4t)$, $r_5(t) = -(1/2)\,(r_1(t) + r_4(t))$, $r_6(t) = -(1/2)\,(r_4(t) + r_3(t))$, $r_7(t) = -(1/2)\,(r_3(t) + r_2(t))$, $r_8(t) = -(1/2)\,(r_1(t) + r_2(t))$, $r_9(t) = -(1/4)(r_1(t) + r_2(t) + r_3(t) + r_4(t))$.

Obviously, the inputs of the agents satisfy the condition in **Definition 1**, i.e., $(1/4)\sum_{j=1}^{4} r_j(t) = -(1/5)\sum_{j=5}^{9} r_j(t)$. Due to the fact that $\lim_{t\to+\infty}\text{arctg}\,(\text{at}) = (\pi/2)$ with $a > 0$, one has $\lim_{t\to+\infty}(r_i(t) - r_j(t)) = 0$ for $\forall\, i, j \in I_1$ or $\forall\, i, j \in I_2$. Furthermore, one can conclude that there do not exist δ and $\xi > 0$ such that the inputs of the agents in the same sub-network satisfies $\lim_{t\to\infty} e^{\xi t}\,(r_i(t) - r_j(t)) = \delta$. Thus, the multi-agent system (1) asymptotically achieves quasi-reverse couple-group consensus. According to (1) with the parameters $\alpha = \beta = 1$, $\gamma = 2$, the evolutions of $x_i(t)$ in sub-network $\Re_1$ and $\Re_2$ are shown in Figs. 4 and 5, respectively. Note that the black solid line represents $(1/4)\sum_{j=1}^{4} r_j(t)$ in Fig. 4, and $(1/5)\sum_{j=5}^{9} r_j(t)$ in Fig. 5, which are the group consensus states, respectively.

In **Theorem 1**, the explicit expression of the error level is derived by $\sigma = c\,\|\Gamma\|\,/\rho$. By some calculations, one obtains $c = \text{arctg}2 - \text{arctg}(1/2) = 0.6435$, $\|\Gamma\| = 12.4067$, and $\rho = 2.8107$, thus $\sigma = 2.8405$. Furthermore, the error of agent i is defined as follows:

$$e_i(t) = \begin{cases} \left| x_i(t) - \dfrac{1}{4}\displaystyle\sum_{j\in I_1} r_j(t) \right|, i \in I_1 \\ \left| x_i(t) - \dfrac{1}{5}\displaystyle\sum_{j\in I_2} r_j(t) \right|, i \in I_2. \end{cases} \tag{28}$$

Then, the evolution of the errors of agent 1 and 5 is shown in Fig. 6. It should be noted that the red dotted lines in Fig. 6 represent the error level $\sigma = 2.8405$, which shows that the estimation of the error level in **Theorem 1** is precise.

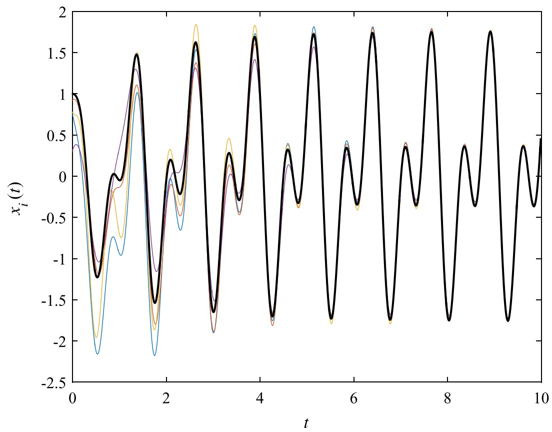

Fig. 7. Evolution of $x_i(t)$ in sub-network $\Re_1$.

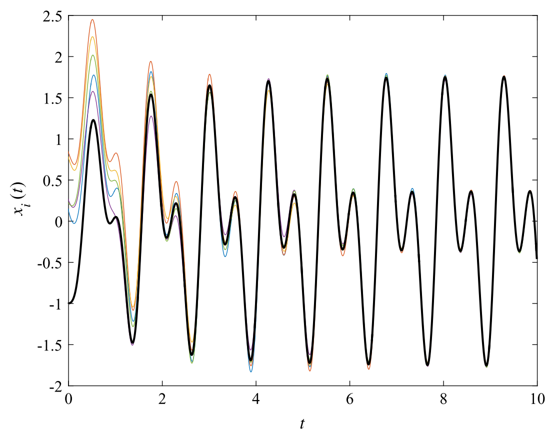

Fig. 8. Evolution of $x_i(t)$ in sub-network $\Re_2$.

Furthermore, to investigate the reverse couple-group consensus, we choose the inputs of the agents as follows:

$$\begin{cases} r_1(t) = \dfrac{e^t - e^{-t}}{e^t + e^{-t}} \sin(10t) + \cos(5t) \\ r_2(t) = \dfrac{e^{0.5t} - e^{-0.5t}}{e^{0.5t} + e^{-0.5t}} \sin(10t) + \cos(5t) \\ r_3(t) = \dfrac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}} \sin(10t) + \cos(5t) \\ r_4(t) = \dfrac{e^{0.25t} - e^{-0.25t}}{e^{0.25t} + e^{-0.25t}} \sin(10t) + \cos(5t) \end{cases} \tag{29}$$

and $r_5(t) = -(1/2)(r_1(t) + r_2(t))$, $r_6(t) = -(1/2)(r_2(t) + r_3(t))$, $r_7(t) = -(1/2)(r_3(t) + r_4(t))$, $r_8(t) = -(1/2)(r_4(t) + r_1(t))$, $r_9(t) = -(1/4)(r_1(t) + r_2(t) + r_3(t) + r_4(t))$.

Then, the inputs of the agents satisfy the condition in **Definition 1**, i.e., $(1/4)\sum_{j=1}^{4} r_j(t) = -(1/5)\sum_{j=5}^{9} r_j(t)$. Moreover, it follows that $\lim_{t\to\infty} e^{0.5t}(r_i(t) - r_j(t))$ exists, for $\forall i, j \in I_1$ or $\forall i, j \in I_2$. Therefore, the multi-agent system (1) asymptotically achieves reverse couple-group consensus, and the evolutions of $x_i(t)$ in sub-network $\Re_1$ and $\Re_2$ are shown in Figs. 7 and 8, respectively. Note that the black solid line represents $(1/4)\sum_{j=1}^{4} r_j(t)$ in Fig. 7, and $(1/5)\sum_{j=5}^{9} r_j(t)$ in Fig. 8, which are the group consensus states, respectively.

V. Conclusion

This paper has discussed the reverse group consensus problem for the dynamic agents with the inputs in the cooperation-competition network. In our framework, the whole network can be divided into two sub-networks, and the weights between the agents in the same sub-network were positive, while the weights between the agents among different sub-networks were negative. Firstly, the reverse group consensus has been studied without the in-degree balance condition presented by the majority of the relevant research works. To solve this problem, the concept of the mirror graph of the cooperation-competition network has been defined, and it has been proven that the reverse group consensus problem can be achieved if the mirror graph is strongly connected. Meanwhile, the explicit expression of the error level has been derived, which would be vanished for special classes of the inputs of the agents. It should be emphasized that the sub-network topology is not required to be strongly connected or contains a directed spanning tree, which means that competition relationship plays a positive role in the reverse group consensus of this paper.

In addition, the decomposing of the cooperation-competition network has been discussed. It has been proven that the cooperation-competition network can be divided into two sub-networks with competition relationship between them if the condensation undirected graph is path balance. Finally, the theoretical results have been verified by representative numerical simulations.

Future work will focus on solving the reverse group consensus of non-linear multi-agent systems in the presence of delayed information communication, as well as switching network topologies.

References

[1] G. Wen, Z. Duan, G. Chen, and W. Yu, "Consensus tracking of multi-agent systems with Lipschitz-type node dynamics and switching topologies," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 2, pp. 499–511, 2014.

[2] M. Cao, F. Xiao, and L. Wang, "Event-based second-order consensus control for multi-agent systems via synchronous periodic event detection," *IEEE Trans. Autom. Control*, vol. 60, no. 9, pp. 2452–2457, 2015.

[3] G. Wen, W. Yu, M. Z. Q. Chen, X. Yu, and G. Chen, "H-infinity pinning synchronization of directed networks with aperiodic sampled-data communications," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 11, pp. 3245–3255, 2014.

[4] W. Xiong, W. Yu, J. Lü, and X. Yu, "Fuzzy modeling and consensus of nonlinear multiagent systems with variable structure," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 4, pp. 1183–1191, 2014.

[5] W. Zhang, S. Liu, and L. Yu, "Fusion estimation for sensor networks with nonuniform estimation rates," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 5, pp. 1485–1498, 2014.

[6] B. Chen, L. Yu, W. Zhang, and A. Liu, "Robust information fusion estimator for multiple delay-tolerant sensors with different failure rates," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 2, pp. 401–414, 2013.

[7] H. Wang and Y. Xie, "Flocking of networked mechanical systems on directed topologies: A new perspective," *Int. J. Control*, vol. 88, no. 4, pp. 872–884, 2015.

[8] G. Wen, Z. Duan, Z. Li, and G. Chen, "Flocking of multi-agent dynamical systems with intermittent nonlinear velocity measurements," *Int. J. Robust. Nonlin. Control*, vol. 22, no. 16, pp. 1790–1805, 2012.

[9] H. Wang, "Flocking of networked uncertain Euler- Lagrange systems on directed graphs," *Automatica*, vol. 49, no. 9, pp. 2774–2779, 2013.

[10] Z. Lin, L. Wang, Z. Han, and M. Fu, "Distributed formation control of multi-agent systems using complex Laplacian," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1765–1777, 2014.

[11] X. Dong, B. Yu, Z. Shi, and Y. Zhong, "Time-varying formation control for unmanned aerial vehicles: Theories and applications," *IEEE Trans. Control Syst. Techol.*, vol. 23, no. 1, pp. 340–348, 2015.

[12] Z. Lin, W. Ding, G. Yan, C. Yu, and A. Giua, "Leader-follower formation via complex Laplacian," *Automatica*, vol. 49, no. 6, pp. 1900–1906, 2013.

[13] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 427–438, 2013.

[14] Y. Chen, J. Lü, X. Yu, and D. Hill, "Multi-agent systems with dynamical topologies: consensus and applications," *IEEE Circuits Syst. Mag.*, vol. 13, no. 3, pp. 21–34, 2013.

[15] J. Yu and L. Wang, "Group consensus of multi-agent systems with switching topologies and communication delays," *Syst. Control Lett.*, vol. 59, no. 6, pp. 340–348, 2010.

[16] H. Hu, L. Yu, W. Zhang, and H. Song, "Group consensus in multi-agent systems with hybrid protocol," *J. Franklin Inst.*, vol. 350, no. 3, pp. 575–597, 2013.

[17] Y. Wang and J. Cao, "Cluster synchronization in nonlinearly coupled delayed networks of non-identical dynamic systems," *Nonlin. Anal. Real World Appl.*, vol. 14, no. 1, pp. 842–851, 2013.

[18] H. Su, Z. Rong, M. Z. Q. Chen, X. Wang, and G. Chen, "Decentralized adaptive pinning control for cluster synchronization of complex dynamical networks," *IEEE Trans. Cybern.*, vol. 43, no. 1, pp. 394–399, 2013.

[19] J. Qin and C. Yu, "Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition," *Automatica*, vol. 49, no. 9, pp. 2898–2905, 2013.

[20] C. Yu, J. Qin, and H. Gao, "Cluster synchronization in directed networks of partial-state coupled linear systems under pinning control," *Automatica*, vol. 50, no. 9, pp. 2341–2349, 2014.

[21] L. Ji, Q. Liu, and X. Liao, "On reaching group consensus for linearly coupled multi-agent networks," *Inform. Sci.*, vol. 287, pp. 1–12, 2014.

[22] H. Hu, A. Liu, Q. Xuan, and L. Yu, "Second-order consensus of multi-agent systems in the cooperation-competition network with switching topologies: A time-delayed impulsive control approach," *Syst. Control Lett.*, vol. 62, no. 12, pp. 1125–1135, 2013.

[23] J. Zhao, Q. Liu, and X. Wang, "Competitive dynamics on complex networks," *Sci. Rep.*, vol. 4, 2014, Art. no. ID5858.

[24] H. Hu, Q. Xuan, W. Yu, C. Zhang, and G. Xie, "Second-order consensus for heterogeneous multi-agent systems in the cooperation-competition network: A hybrid adaptive and pinning control approach," *Nonlin. Anal. Hybrid Syst.*, vol. 20, pp. 21–36, 2016.

[25] G. Chen, "Pinning control and synchronization on complex dynamica networks," *Int. J. Control Autom. Syst.*, vol. 12, no. 2, pp. 21–36, 2014.

[26] X. Wang and H. Su, "Pinning control of complex networked systems: A decade after and beyond," *Annu. Rev. Control*, vol. 38, no. 1, pp. 103–111, 2014.

[27] M. Valcher and P. Misra, "On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions," *Syst. Control Lett.*, vol. 66, no. 1, pp. 94–103, 2014.

[28] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, 2013.

[29] C. Altafini, "Stability analysis of diagonally equipotent matrices," *Automatica*, vol. 49, no. 9, pp. 2780–2785, 2013.

[30] C. Altafini and G. Lini, "Predictable dynamics of opinion forming for networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 60, no. 2, pp. 342–357, 2015.

[31] D. Meng, M. Du, and Y. Jia, "Interval bipartite consensus of networked agents associated with signed digraphs," *IEEE Trans. Autom. Control*, to be published, doi: 10.1109/TAC.2016.2528539.

[32] S. Kia, J. Cortés, and S. Martínez, "Distributed event-triggered communication for dynamic average consensus in networked systems," *Automatica*, vol. 59, pp. 103–111, 2015.

[33] S. Kia, J. Cortés, and S. Martínez, "Dynamic average consensus under limited control authority and privacy requirements," *Int. J. Robust. Nonlin. Control*, vol. 25, no. 13, pp. 1941–1966, 2015.

[34] W. Yu, G. Chen, M. Cao, and W. Ren, "Delay-induced consensus and quasi-consensus in multi-agent dynamical systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 10, pp. 2679–2687, 2013.

[35] W. He, F. Qian, Q. Han, and J. Cao, "Lag quasi-synchronization of coupled delayed systems with parameter mismatch," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 6, pp. 1345–1357, 2011.

[36] W. He, F. Qian, J. Lam, G. Chen, Q. Han, and J. Kurths, "Quasi-synchronization of heterogeneous dynamic networks via distributed impulsive control: error estimation, optimization and design," *Automatica*, vol. 62, pp. 249–262, 2015.

Hong-xiang Hu received the B.S. degree in electronic engineering from Zhejiang Gongshang University in 2005, the M.S. degree in mathematics from Hangzhou Dianzi University in 2009, and the Ph.D. degree in control theory and control engineering from Zhejiang University of Technology in 2013.

Since September 2015, he has been postdoctoral researcher with Southeast University. Currently, he is a Lecturer at Hangzhou Dianzi University, Hangzhou, China. His research interests include multiagent systems, collective dynamics, complex networks, hybrid and switched systems, and cyber-physical systems.

Wenwu Yu (S'07–M'12–SM'15) received the B.Sc. degree in information and computing science and the M.Sc. degree in applied mathematics from the Department of Mathematics, Southeast University, Nanjing, China, in 2004 and 2007, respectively, and the Ph.D. degree from the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, in 2010.

Currently, he is the Founding Director of the Laboratory of Cooperative Control of Complex Systems, an Associate Director in the Research Center for Complex Systems and Network Sciences, an Associate Dean in the Department of Mathematics, and a Full Professor with the Distinguished Young Honor at Southeast University. He held several visiting positions in Australia, China, Germany, Italy, the Netherlands, and the U.S. His research interests include multi-agent systems, complex networks and systems, nonlinear dynamics and control, smart grids, neural networks, game theory, robotics, and optimization.

Dr. Yu was listed by Thomson Reuters Highly Cited Researchers in Engineering in 2014 and 2015, respectively. He was awarded a National Natural Science Fund for Excellent Young Scholars in 2013, the Six Talent Peaks of Jiangsu Province of China in 2014, and the National Ten Thousand Talent Program for Young Top-notch Talents in 2015.

Guanghui Wen (S'11–M'13) received the Ph.D. degree in mechanical systems and control from Peking University, Beijing, China, in 2012.

From 2012 to 2013, he was a Research Associate and a Post-Doctoral Fellow with the University of New South Wales at Canberra, Campbell, ACT, Australia. In 2014, he was a Visiting Research Fellow with the School of Electrical and Computer Engineering, Royal Melbourne Institute of Technology University, Melbourne, VIC, Australia, for three months. He is currently an Associate Professor with the Department of Mathematics, Southeast University, Nanjing, China. His current research interests include cooperative control of multiagent systems, networked control systems, robust control, and cyber-physical systems.

Dr. Wen was a recipient of the Best Student Paper Award from the 6th Chinese Conference on Complex Networks in 2010. He is an Associate Editor of the *Asian Journal of Control*.

Qi Xuan received the B.Eng. and Ph.D. degrees in control theory and engineering from Zhejiang University, Hangzhou, China, in 2003 and 2008, respectively.

He was a Postdoctoral Researcher in the Department of Information Science and Electronic Engineering, Zhejiang University, from 2008 to 2010, and was a Research Assistant in the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, in 2010. From 2012 to 2014, he was a Postdoctoral Researcher in the Department of Computer Science, University of California at Davis. Currently, he is an Associate Professor in the College of Information Engineering, Zhejiang University of Technology. His research interests include network based algorithm design, social network data mining, social synchronization and consensus, reaction-diffusion network dynamics, machine learning, and computer vision.

Jinde Cao (M'07–SM'07–F'16) is a Distinguished Professor, the Dean of Department of Mathematics and the Director of the Research Center for Complex Systems and Network Sciences, Southeast University, Nanjing, China. From March 1989 to May 2000, he was with Yunnan University. In May 2000, he joined the Department of Mathematics, Southeast University. From July 2001 to June 2002, he was a Postdoctoral Research Fellow at the Department of Automation and Computer-Aided Engineering, Chinese University of Hong Kong, Hong Kong.

Dr. Cao was an Associate Editor of the IEEE TRANSACTIONS ON NEURAL NETWORKS, the *Journal of the Franklin Institute* and *Neurocomputing*. He is Associate Editor of the IEEE TRANSACTIONS ON CYBERNETICS, *Differential Equations and Dynamical Systems*, *Mathematics and Computers in Simulation*, and *Neural Networks*. He is a Member of the Academy of Europe. He has been named as Highly-Cited Researcher in *Mathematics*, *Computer Science* and *Engineering* by Thomson Reuters.