Reaction-diffusion processes and metapopulation models on duplex networks

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Reaction-diffusion processes, used to model various spatially distributed dynamics such as epidemics, have been studied mostly on regular lattices or complex networks with simplex links that are identical and invariant in transferring different kinds of particles. However, in many self-organized systems, different particles may have their own private channels to keep their purities. Such division of links often significantly influences the underlying reaction-diffusion dynamics and thus needs to be carefully investigated. This article studies a special reaction-diffusion process, named susceptible-infected-susceptible (SIS) dynamics, given by the reaction steps $\beta \to \alpha$ and $\alpha + \beta \to 2\beta$, on duplex networks where links are classified into two groups: α and β links used to transfer α and β particles, which, along with the corresponding nodes, consist of an α subnetwork and a β subnetwork, respectively. It is found that the critical point of particle density to sustain reaction activity is independent of the network topology if there is no correlation between the degree sequences of the two subnetworks, and this critical value is suppressed or extended if the two degree sequences are positively or negatively correlated, respectively. Based on the obtained results, it is predicted that epidemic spreading may be promoted on positive correlated traffic networks but may be suppressed on networks with modules composed of different types of diffusion links.

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I. INTRODUCTION

The world is fully filled with different levels of particles, such as molecules, cells, organisms, even people and computer documents, etc. These particles diffuse in certain spaces through random walks or by definite forces. At the same time, reactions take place when particles on the same level meet in a local world and are transformed into each other with certain probabilities. Over the past few decades, such reaction-diffusion (RD) systems have been studied extensively [1–3], providing good explanations for many spatially distributed dynamics existing in various areas, such as chemical reactions [4], biological pattern formation [5], population evolution [6], epidemics [7], computer virus spreading [8], and so on. Observing that a wide variety of phenomena involve coupling effects, e.g., in coupled chemical reactors [9], protein-protein networks [10], neural networks [11], social networks [12], etc., researchers have paid more and more attention to RD dynamics of spatially discrete systems, which are relatively easier to be realized by advanced computer technologies.

Recent studies have revealed that many real-world complex networks possess heterogeneous structures characterized by power-law degree distributions [13–15], where the degree of a node is defined as the number of its directly connected neighbors, attributed first to Barabási and Albert [13]. Further experiments [16–18] demonstrated that RD dynamics on such heterogeneous networks may behave totally differently from those on regular lattices, which suggests that RD dynamics are not only determined by the reaction equations involved, but they are also significantly influenced by the structures of the diffusion spaces. For example, it was found that epidemic described by the well-known susceptible-infected-susceptible (SIS) model [19] may be sustained on heterogeneous networks even with a vanishing density of individuals, which, however, is not so on regular lattices. This is not surprising because most of these studies on RD processes adopt a common hypothesis: every link in a diffusion network can be used to transfer all kinds of particles, although sometimes different kinds of particles are assumed to have different diffusion rates. This hypothesis ensures that different kinds of particles, uniformly distributed initially, will be concentrated on several nodes of large degrees in a heterogeneous network through a random walk process, and thus will significantly improve the reaction activity.

Such simplification is reasonable in disordered systems where each particle is allowed to diffuse in all directions with the same probability. However, it is more practical to consider those self-organized systems in which different kinds of particles may have their preferences to be transferred through certain private channels. For example, it was revealed that national highways and interprovincial freeways played more important roles than railways to spread severe acute respiratory syndrome (SARS) in mainland China by analyzing the cases in 345 counties of three provinces near Beijing, i.e., Hebei, Shanxi, and the Inner Mongolia Autonomous Region [20]. This indicates that many people preferred to use buses or cars to return home when the epidemic broke out in Beijing, because these means of transportation are more flexible in time and thus also made the passengers less supervised. Meanwhile, Askar *et al.* [21] found that high-efficiency particulate air (HEPA) filtration is frequently used in airplanes but not in ground transport vehicles, which may lead to higher disease transmission through ground transport systems than airplanes. Besides, microscopically, an infected individual may infect others by different types of contact with different probabilities, as pointed out by Stegeman *et al.* when they studied interherd

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transmission of classical swine fever virus during the 1997–1998 epidemic in the Netherlands [22].

In order to simplify the problem and emphasize the preferences of links on transferring certain kinds of particles, here we just consider the situation in which links are specialized to transfer certain kinds of particles. In particular, when considering SIS dynamics on a duplex network with variant links, i.e., α and β links used to transfer α and β particles, respectively, these two kinds of particles may be concentrated on different central nodes with different probabilities, depending on the correlation between the degree sequences of the specialized subnetworks used for transferring different kinds of particles. This is quite different from the situation on heterogeneous networks with identical links. Consequently, such division of links in heterogeneous networks may further influence RD dynamics and thus has to be carefully investigated so as to better understand the many complex phenomena in real-world self-organized systems.

The rest of the paper is organized as follows. In Sec. II, a special RD process described by the basic SIS model on simplex networks is reviewed, which is then studied on duplex networks composed of two types of links. In Sec. III, some analytic results about critical points of the phase transition from an absorbing phase to an active phase of the RD process on duplex networks are obtained. Then the RD process is studied on duplex scale-free networks and duplex networks with modules in Secs. IV and V, respectively, where a series of analytic and simulated results are provided. The work is finally concluded in Sec. VI.

II. REACTION-DIFFUSION PROCESSES ON DUPLEX NETWORKS

First, a basic reaction mechanism involving only two kinds of particles is reviewed, which is composed of the following two reactions:

$$\beta \to \alpha, \tag{1}$$

$$\alpha + \beta \to 2\beta. \tag{2}$$

This scheme is also known as the SIS model [23] and has been extensively studied in physics and mathematical epidemiology [18,24–26], where α particles represent normal particles (healthy individuals) and β particles represent active particles (infected individuals). Since the particles are neither regenerated nor disappeared, but only transformed from one to another, the total number of particles does not change in the process. Generally, α and β particles at each node react with each other according to Eqs. (1) and (2) at reaction rates μ_1 and μ_2, respectively. After the reaction process, the particles undergo a diffusion process as follows. Each particle on node v_i with degree k_i jumps into one of its neighbors with probability $1/k_i$, and suppose that the two kinds of particles have the same unitary diffusion rates. In this case, by the mean-field theory, it was shown [19] that the transition from an absorbing phase (no β particles) to an active phase (with sustained β particles) occurs when $\rho = \rho_c = \mu_1/(H\mu_2)$, where $H = \langle k^2\rangle/\langle k\rangle^2$ is the heterogeneity of the network. Since it is always satisfied that $\langle k^2\rangle \geqslant \langle k\rangle^2$ for all networks, i.e., $H \geqslant 1$, where $\langle k^2\rangle = \langle k\rangle^2$ is obtained only when each pair of nodes has the same degree, the critical point will never be larger than μ_1/μ_2 on the networks with simplex links.

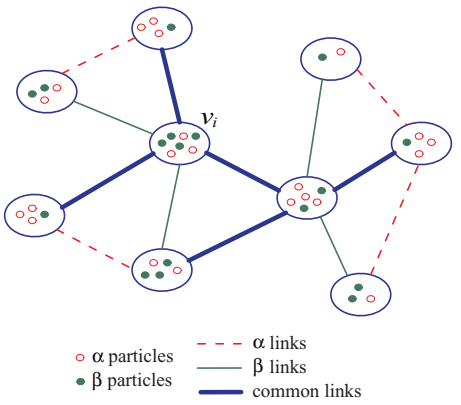

FIG. 1. (Color online) Suppose that an RD process involves α and β particles. A duplex RD network has two types of links, i.e., α links (dashed red lines) to transfer α particles only, and β links (solid green lines) to transfer β particles only. If pairwise nodes are linked by both α and β links, it is simply considered that they are linked by common links (bold blue lines) to transfer α and β particles together. The types of lines for the three kinds of links keep the same in the following network figures.

In this paper, the SIS model is studied on a duplex network with two groups of links: α links to transfer α particles and β links to transfer β particles, which, along with the corresponding nodes, consist of an α subnetwork and a β subnetwork, respectively. Note that a pair of nodes may be linked by both α and β links; in such a situation, it is simply considered that they are linked by a common link which can transfer both α and β particles. Denote by $k_{\alpha,i}$ and $k_{\beta,i}$ the α degree and β degree of node v_i, representing the numbers of its neighbors in the α subnetwork and β subnetwork, respectively. For example, the node v_i in the duplex RD network shown in Fig. 1 has $k_{\alpha,i} = 3$ and $k_{\beta,i} = 5$. In each node, the spontaneous process $\beta \to \alpha$ simply consists of turning each β particle into an α particle with rate μ_1. Since a type-I reaction introduced in [19] is considered here, the $\alpha + \beta \to 2\beta$ process means that each α particle may react with all of the β particles in the same node and each reaction occurs with rate μ_2. Note that here the diffusion rates of α particles and β particles are set to be the same, i.e., $\eta = 1$, for simplicity, which means that all the α particles (or β particles) in a node at present will jump into its neighboring nodes in the α subnetwork (or β subnetwork) at the next time step. Moreover, the numbers of α and β particles at a node can be any non-negative integers, known as the bosonic process [18,19].

Suppose that there are $V(k_\alpha,k_\beta)$ nodes of α-degree k_α and β-degree k_β in the network, which follow a joint distribution $P(k_\alpha,k_\beta)$. Denoted by $\rho_\alpha(k_\alpha,k_\beta)$ and $\rho_\beta(k_\alpha,k_\beta)$, the average densities of α particles and β particles, respectively, among these nodes, satisfy

$$\rho_\alpha(k_\alpha,k_\beta) = \frac{N_\alpha(k_\alpha,k_\beta)}{V(k_\alpha,k_\beta)}, \quad \rho_\beta(k_\alpha,k_\beta) = \frac{N_\beta(k_\alpha,k_\beta)}{V(k_\alpha,k_\beta)}, \tag{3}$$

with $N_\alpha(k_\alpha,k_\beta)$ and $N_\beta(k_\alpha,k_\beta)$ denoting the total numbers of α and β particles, respectively, in these nodes. The average densities of α and β particles of the whole network

then is given by $\rho_\alpha = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\rho_\alpha(k_\alpha,k_\beta)$ and $\rho_\beta = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\rho_\beta(k_\alpha,k_\beta)$, respectively. The quantities of $\rho_\alpha(k_\alpha,k_\beta)$ and $\rho_\beta(k_\alpha,k_\beta)$ vary with respect to the nodes of different α degrees and β degrees, and their time evolution can be expressed by a set of ordinary differential equations depending on the reaction kernel defined by $\Omega(k_\alpha,k_\beta) = \rho_\alpha(k_\alpha,k_\beta)\rho_\beta(k_\alpha,k_\beta)$. Then the critical point of the phase transition can be obtained by solving these equations.

In particular, RD processes on duplex networks are carried out by the Monte Carlo method as follows:

(1) *Initialization*. Numbers $V\rho_\alpha(0)$ of α particles and $V\rho_\beta(0)$ of β particles are randomly distributed on a network containing V nodes, with a particle density $\rho = \rho_\alpha(0) + \rho_\beta(0)$. The results are independent of the particular initial ratio $\rho_\alpha(0)/\rho_\beta(0)$, apart from early time transients.

(2) *Reaction*. At each time t, numbers $n_{\alpha,i}(t-1)$ of α particles and $n_{\beta,i}(t-1)$ of β particles in each node v_i react with each other according to Eqs. (1) and (2). That is, each β particle in node v_i is transformed to an α particle with probability μ_1 and the reverse takes place with probability $1-(1-\mu_2)^{n_{\beta,i}(t-1)}$ [19]. This corresponds to the average probability of an α particle being involved in Eq. (2) with any of the β particles on the same node. Here, $n_{\alpha,i}(t-1)$ and $n_{\beta,i}(t-1)$ can be any non-negative integers including zero when considering the bosonic RD process [18].

(3) *Diffusion*. After reactions, every α (or β) particle in each node of α-degree k_α (or β-degree k_β) jumps into one of its neighbors in an α subnetwork (or β subnetwork) with the same probability $1/k_\alpha$ (or $1/k_\beta$). Then, after a round of RD process, the numbers of α particles and β particles in node v_i are updated to be $n_{\alpha,i}(t)$ and $n_{\beta,i}(t)$, respectively.

III. CRITICAL POINTS OF THE PHASE TRANSITION

Suppose there are no degree correlations between linked nodes in the α subnetwork and β subnetwork, respectively. Then, by the mean-field theory [27], the stationary states of the system are given by (see Appendix A)

$$\rho_\alpha(k_\alpha,k_\beta) = \frac{k_\alpha}{\langle k_\alpha\rangle}[\mu_1\rho_\beta + \rho_\alpha - \mu_2\Omega], \tag{4}$$

$$\rho_\beta(k_\alpha,k_\beta) = \frac{k_\beta}{\langle k_\beta\rangle}[(1-\mu_1)\rho_\beta + \mu_2\Omega], \tag{5}$$

with $\Omega = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\Omega(k_\alpha,k_\beta)$. Equations (4) and (5) suggest that the densities of α particles and β particles at nodes are proportional to their α degree and β degree, respectively. Since the generation of β particles is influenced by the product of the local densities of α particles and β particles, more active particles can be observed on heterogeneous networks only when their α-degree and β-degree sequences are positively correlated, i.e., nodes with larger α degrees also possess larger β degrees.

Denoting by ρ the average density of the two kinds of particles in the network, satisfying $\rho = \rho_\alpha + \rho_\beta$, then the critical point ρ of the phase transition to sustain β particles is obtained as (see Appendix A)

$$\rho_c = \frac{\mu_1}{\mu_2}\frac{\langle k_\alpha\rangle\langle k_\beta\rangle}{\langle k_\alpha k_\beta\rangle} = \frac{\mu_1}{\mu_2}\frac{V\langle k_\alpha\rangle\langle k_\beta\rangle}{S_\alpha S_\beta{}^T}, \tag{6}$$

where $S_\alpha = [k_{\alpha,1},k_{\alpha,2},\ldots,k_{\alpha,V}]$ and $S_\beta = [k_{\beta,1},k_{\beta,2},\ldots,k_{\beta,V}]$ are the α-degree and the corresponding β-degree sequences, respectively, of the network. This indicates that the critical point equals μ_1/μ_2, independent of the network structure, when there is no correlation between the two degree sequences. Moreover, the value of ρ_c decreases if the two degree sequences are positively correlated but increases if they are negatively correlated. Then ρ_c is minimized or maximized just by varying the correlation between the α-degree and β-degree sequences, if other parameters are unchangeable, i.e., ρ_c is minimized if the two sequences are in the same order, while it is maximized if one of them is inversely ordered. Moreover, when the α degree and β degree of each node are changeable in the ranges $[L_\alpha,U_\alpha]$ and $[L_\beta,U_\beta]$, respectively, satisfying that the average α degree and β degree are fixed, the minimum critical point is obtained when the two degree sequences take on the following forms:

$$S_\alpha = [L_\alpha,\ldots,L_\alpha,k_{\alpha,\nu},U_\alpha,\ldots,U_\alpha], \tag{7}$$

$$S_\beta = [L_\beta,\ldots,L_\beta,k_{\beta,\omega},U_\beta,\ldots,U_\beta], \tag{8}$$

while the critical point is maximized by inversely ordering only one of the degree sequences presented in Eqs. (7) and (8).

Suppose that the α degree and β degree of each node are in the same range $[L,U]$ and the α subnetwork and β subnetwork have the same average degree, i.e., $\langle k_\alpha\rangle = \langle k_\beta\rangle = R$. If $U \gg R > L$, then the minimum and maximum critical points are estimated by

$$\rho_{c,\min} \approx \frac{\mu_1}{\mu_2}\frac{R^2}{U(R-L)}, \tag{9}$$

$$\rho_{c,\max} \approx \frac{\mu_1}{\mu_2}\frac{R^2}{2LR-L^2}, \tag{10}$$

respectively (see Appendix B). It should be noted that these results are obtained under the assumption of no degree correlations in the two subnetworks. However, this assumption cannot be held any longer if the lower bound of degree L is very small (i.e., close to 1), i.e., both the α subnetwork and β subnetwork have a similar quasistar structure, and thus the above results need to be further modified. The two quasistar subnetworks are merged into a unique quasistar network, as shown in Fig. 2(c), if the α-degree and β-degree sequences are positively correlated, while they are merged into a network having two modules with the nodes in each module being connected by the same type of links and the nodes of different modules being connected by common links, as shown in Fig. 2(d), if the two degree sequences are negatively correlated.

Since both the networks are composed of two classes of nodes, i.e., kernel nodes and leaf nodes in the quasistar network and nodes of different modules in the modular network, and the nodes of the same class are completely identical, it is equivalent to study the RD processes on networks with only two coupled nodes, denoted by A and B, as shown in Figs. 2(e) and 2(f), respectively. Suppose there are the same number of nodes in the two classes. Then, in case I, the diffusion rate of particles from B to A is equal to 1, since all the particles in the leaf nodes at present will jump into kernel nodes at the next time step based on the assumption that α and β particles have the same diffusion rate $\eta = 1$ on the initial diffusion network, while that from A to B is just equal to $\theta = L/U$, that is, only

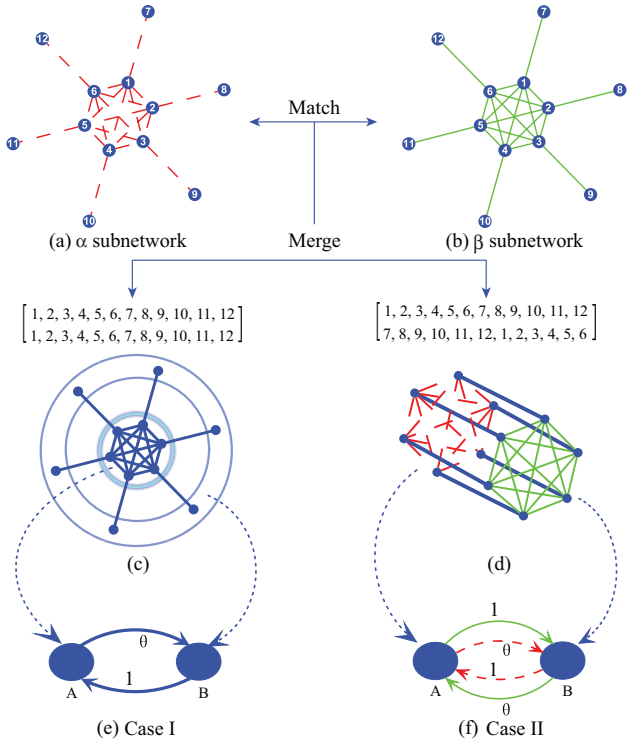

FIG. 2. (Color online) A duplex RD network can be obtained by merging different subnetworks connected through the same types of links after a node matching process. Suppose there are two subnetworks: (a) quasistar α-subnetwork and (b) quasistar β-subnetwork. They are merged into (c) a unique quasistar network where each link can transfer both α and β particles, if the α-degree and β-degree sequences are positively correlated, while they are merged into (d) a network with two modules, if the two degree sequences are negatively correlated. Both the networks are composed of two classes of nodes, i.e., kernel nodes and leaf nodes in the quasistar network and nodes of different modules in the modular network. The nodes of same class can be considered as a supernode since they are completely identical. Then, alternatively, we only need to study the RD processes on networks with two coupled nodes in (e) case I and (f) case II, respectively.

a small fraction of particles in the kernel nodes will jump into the leaf nodes at each time step. The situation in case II is a little more complicated, where the diffusion rate of β particles from A to B and that of α particles from B to A are equal to 1, while the diffusion rate of α particles from A to B and that of β particles from B to A are equal to $\theta = L/U$, as shown in Fig. 2(f). By solving the RD processes on such directed networks with only two coupled nodes, Eqs. (9) and (10) are modified as (see Appendix C)

$$\rho_{c,\min} = \frac{\mu_1}{\mu_2}\frac{1+\theta}{2}, \tag{11}$$

$$\rho_{c,\max} = \left(1+\frac{1}{\theta}\right)\left[\frac{\sqrt{4+\mu_1^2(1-\theta)^2}+\mu_1(1+\theta)-2}{4\mu_2}\right]. \tag{12}$$

As $\theta \to 0$, Eq. (11) predicts that $\rho_{c,\min} \to \mu_1/2\mu_2$, which is the same as that obtained by Eq. (9), considering that here $\theta = L/U$ and $R = (L+U)/2$. However, Eq. (12) can be rewritten as

$$\rho_{c,\max} = \frac{\mu_1}{\mu_2}\frac{(1+\theta)^2}{4\theta} + \frac{(1+\theta)\left[\sqrt{4+\mu_1^2(1-\theta)^2}-2\right]}{4\theta\mu_2}, \tag{13}$$

where the value calculated by the first term is equal to that obtained by Eq. (10) and thus the maximum critical point derived by Eq. (12) is always larger than that predicted by Eq. (10), as θ tends to zero.

Since many real-world networks possess random rather than deterministic structures [28], next, RD processes will be studied on more realistic random networks to investigate the effect of α-degree and β-degree correlation on the phase transition.

IV. PHASE TRANSITION ON DUPLEX SCALE-FREE NETWORKS

It has been revealed that many real-world networks including traffic networks are scale-free networks characterized by a power-law degree distribution, i.e., $P(k) \sim k^{-\gamma}$. Since different types of transportation systems may have different levels of risk to spread epidemics, it is becoming crucial to study the phase transition of SIS dynamics on duplex scale-free networks. In this paper, duplex scale-free networks with different correlations between α-degree and β-degree sequences are created by the following three steps:

(1) *Initialization*. Two independent networks G_α and G_β with same number of nodes are created by the uncorrelated random scale-free network model proposed by Catanzaro *et al.* [29], which are determined by three parameters, i.e., the number of nodes V, the lower bound of degrees m, and the exponent of degree distribution γ. Meanwhile, the list of nodes in each network is ordered by their degrees, i.e., $k_{\alpha,i} \geqslant k_{\alpha,j}$ and $k_{\beta,i} \geqslant k_{\beta,j}$ if $i > j$ for $i,j = 1,2,\ldots,V$.

(2) *Matching*. Denote by $\{v_{\alpha,1}, v_{\alpha,2}, \ldots, v_{\alpha,V}\}$ and $\{v_{\beta,1}, v_{\beta,2}, \ldots, v_{\beta,V}\}$ the lists of ordered nodes in the networks G_α and G_β, respectively. With probability p, node $v_{\alpha,i}$ in the network G_α is matched to $v_{\beta,V-i}$ in the network G_β; otherwise, it is matched to $v_{\beta,i}$.

(3) *Merging*. A new duplex RD network G_p, with α- and β-degree correlation equal to $1-p$, is established by merging each pairwise matched nodes as one node and inheriting the associated α and β links.

Based on this mechanism, the α-degree and β-degree sequences of the duplex RD network G_p tend to be positively correlated as $p \to 0$ and tend to be negatively correlated as $p \to 1$. For example, Fig. 3 provides several such duplex RD networks with different values of p.

In order to validate the analytic result represented by Eq. (6), we do a number of simulations on duplex scale-free RD networks with different sizes and different correlations between α-degree and β-degree sequences. Here, the parameters to create the initial networks G_α and G_β are set to be $m = 2$, $\gamma = 2$, and $V = 500, 1000, 2000$, respectively. For each value of p, $p = 0, 0.1, 0.2, \ldots, 1$, and each network size, a corresponding RD network is obtained. In the beginning of the RD process, the same numbers of α particles and β particles are randomly distributed on the RD network. When the RD

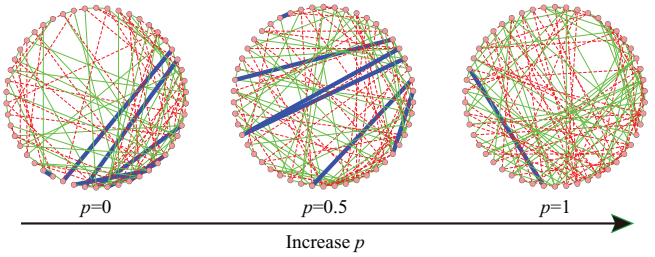

FIG. 3. (Color online) The process to generate duplex RD networks with different correlations between α-degree and β-degree sequences, which is controlled by the parameter p. Here, the parameters to create the initial networks G_α and G_β are set to be $m = 2$, $\gamma = 2$, $V = 50$.

process on the network is relatively steady after sufficient time steps (10^6 here), it is implemented in parallel an extra 10^4 time steps with the reaction rates $\mu_1 = 0.1$ and $\mu_2 = 0.05$. Then, the average density of β particles in this 10^4 times as well as the variance is recorded.

The phase transitions of RD processes on duplex scale-free networks with $V = 1000$ nodes and different $p = 0,0.2,0.5,0.8,1$ are shown in Fig. 4(a), where we can see that the process undergoes a phase transition $\rho_c < \mu_1/\mu_2 = 2$ as the α-degree and β-degree sequences of the duplex RD network tend to be positively correlated ($p \to 0$), while it undergoes a phase transition $\rho_c > \mu_1/\mu_2 = 2$ as the two degree sequences of the duplex RD network tend to be negatively correlated ($p \to 1$). This result is consistent with our theoretical analysis. In fact, for heterogeneous networks, positive correlation between two degree sequences means that the nodes with high α degree always have high β degree, so that most α and β particles contact with each other in these dual-hub nodes. In such a situation, the phase transition to sustain β particles is suppressed, since there are always a small number of nodes with enough α and β particles to keep the RD process alive, even for a vanishing average density of particles. On the other hand, when the two degree sequences are negatively correlated, the hubs in the α subnetwork and those in the β subnetwork are separated, which decreases the probability that α and β particles contact each other. Therefore a quite high average density of particles is needed to generate β particles, leading to an extension of the phase transition.

In order to provide more quantitative comparison between analytic and simulated results, we also calculate the analytic critical points of phase transitions on duplex scale-free networks with different size by Eq. (6), and meanwhile estimate the corresponding simulated critical points from simulation results. In particular, the analytic and simulated critical points as functions of p for duplex scale-free networks including different numbers of nodes, i.e., $V = 500$, $V = 1000$, and $V = 2000$, are shown in Figs. 4(b)–4(d), respectively, where we find that the analytic and simulated critical points have the same trend as p increases. It should be noted that the analytic results are obtained by applying the mean-field theory with the assumption that the network size is sufficiently large; as a result, there may be a slight gap between the real critical point and that obtained by the mean-field theory when the network

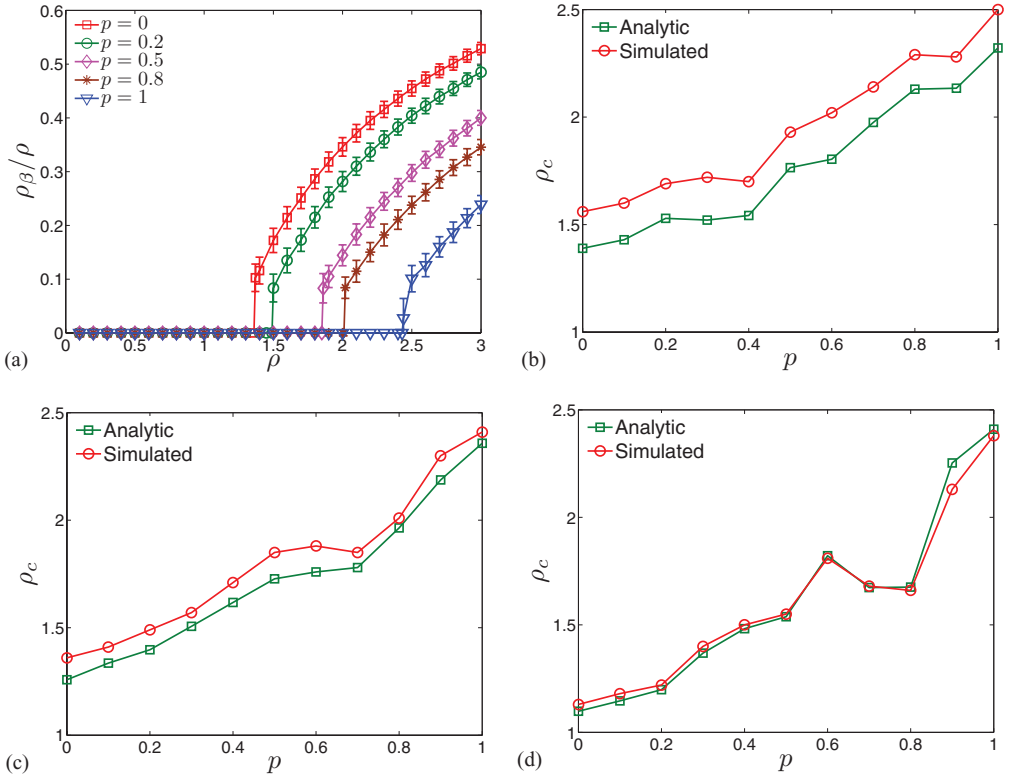

FIG. 4. (Color online) (a) Phase transitions for RD processes on duplex scale-free networks with $m = 2$, $\gamma = 2$, $V = 1000$, and different $p = 0,0.2,0.5,0.8,1$. (b)–(d) The analytic and simulated critical points as functions of p for duplex scale-free networks including different numbers of nodes, i.e., $V = 500$, $V = 1000$, and $V = 2000$, respectively.

size is not large enough, as shown in Fig. 4(b), which will disappear gradually as the network size increases, as shown in Figs. 4(c) and 4(d). Denote by $\rho_c^A(p)$ and $\rho_c^S(p)$ the analytic and simulated critical points of phase transition on the duplex network with parameter p and define the relative difference between $\rho_c^A(p)$ and $\rho_c^S(p)$ by

$$\epsilon(p) = \frac{\left|\rho_c^A(p) - \rho_c^S(p)\right|}{\rho_c^A(p)} \times 100\%. \tag{14}$$

Then, for N_p cases of different p, the average relative difference between the analytic and simulated critical points is calculated by

$$\epsilon = \frac{\sum_p \epsilon(p)}{N_p}. \tag{15}$$

Here, we have $N_p = 11$, and such average relative difference ϵ is equal to 10.0%, 5.3%, and 1.9% for the duplex scale-free RD networks with $V = 500$, $V = 1000$, and $V = 2000$, respectively, which indicates that Eq. (6) predicts the critical point of the RD process on uncorrelated random scale-free networks very well when the network size is sufficiently large.

V. PHASE TRANSITION ON DUPLEX NETWORKS WITH MODULES

Since many real-world networks contain communities [30,31] or motifs [32,33], it is of much interest to study RD processes on duplex networks with modular structures. Consider the RD network consisting of two modules denoted by M_α and M_β, with V_α and V_β nodes, respectively. Each pair of nodes in M_α or in M_β is connected by an α link or a β link with probability p_α or p_β, respectively, and the nodes from different modules are connected by a common link with probability p_b. On average, the α degree and β degree of a node in M_α are calculated by

$$k_{\alpha,\alpha} = p_\alpha(V_\alpha - 1) + p_b V_\beta, \quad k_{\beta,\alpha} = p_b V_\beta, \tag{16}$$

while those in M_β are calculated by

$$k_{\alpha,\beta} = p_b V_\alpha, \quad k_{\beta,\beta} = p_\beta(V_\beta - 1) + p_b V_\alpha. \tag{17}$$

Then the average α degree, the average β degree, and the average correlation between them in the RD network are obtained as

$$\langle k_\alpha \rangle = \frac{V_\alpha k_{\alpha,\alpha} + V_\beta k_{\alpha,\beta}}{V_\alpha + V_\beta} = \frac{p_\alpha V_\alpha(V_\alpha - 1) + 2p_b V_\alpha V_\beta}{V_\alpha + V_\beta}, \tag{18}$$

$$\langle k_\beta \rangle = \frac{V_\alpha k_{\beta,\alpha} + V_\beta k_{\beta,\beta}}{V_\alpha + V_\beta} = \frac{p_\beta V_\beta(V_\beta - 1) + 2p_b V_\alpha V_\beta}{V_\alpha + V_\beta}, \tag{19}$$

$$\langle k_\alpha k_\beta \rangle = \frac{V_\alpha k_{\alpha,\alpha} k_{\beta,\alpha} + V_\beta k_{\alpha,\beta} k_{\beta,\beta}}{V_\alpha + V_\beta} = \frac{p_b V_\alpha V_\beta[p_\alpha(V_\alpha - 1) + p_b V_\beta] + p_b V_\alpha V_\beta[p_\beta(V_\beta - 1) + p_b V_\alpha]}{V_\alpha + V_\beta}, \tag{20}$$

respectively.

A. Dominant module M_α ($V_\alpha \gg V_\beta > 0$)

In this case, Eqs. (18)–(20) are simplified to

$$\begin{aligned} \langle k_\alpha \rangle &= p_\alpha V_\alpha, \quad \langle k_\beta \rangle = 2p_b V_\beta, \\ \langle k_\alpha k_\beta \rangle &= p_\alpha p_b V_\alpha V_\beta + {p_b}^2 V_\alpha V_\beta. \end{aligned} \tag{21}$$

Substituting these values into the expression of ρ_c, represented by Eq. (6), the critical point of the phase transition to generate active particles is obtained as

$$\rho_c = \frac{\mu_1}{\mu_2} \frac{2p_\alpha}{p_\alpha + p_b}. \tag{22}$$

B. Dominant module M_β ($V_\beta \gg V_\alpha > 0$)

In this case, Eqs. (18)–(20) are simplified to

$$\begin{aligned} \langle k_\alpha \rangle &= 2p_b V_\alpha, \quad \langle k_\beta \rangle = p_\beta V_\beta, \\ \langle k_\alpha k_\beta \rangle &= p_\beta p_b V_\alpha V_\beta + {p_b}^2 V_\alpha V_\beta. \end{aligned} \tag{23}$$

Substituting these values into Eq. (6), the critical point is obtained as

$$\rho_c = \frac{\mu_1}{\mu_2} \frac{2p_\beta}{p_\beta + p_b}. \tag{24}$$

C. Modules of the same size ($V_\alpha = V_\beta = V/2$)

In this case, we have

$$\begin{aligned} \langle k_\alpha \rangle &= \frac{V}{4}(p_\alpha + 2p_b), \quad \langle k_\beta \rangle = \frac{V}{4}(p_\beta + 2p_b), \\ \langle k_\alpha k_\beta \rangle &= \frac{V^2}{8}(p_\alpha p_b + p_\beta p_b + 2{p_b}^2). \end{aligned} \tag{25}$$

Then the critical point is calculated by

$$\begin{aligned} \rho_c &= \frac{\mu_1}{\mu_2} \frac{V^2(p_\alpha + 2p_b)(p_\beta + 2p_b)}{2V^2\left(p_\alpha p_b + p_\beta p_b + 2p_b^2\right)} \\ &= \frac{\mu_1}{\mu_2}\left(1 + \frac{p_\alpha p_\beta}{2p_\alpha p_b + 2p_\beta p_b + 4{p_b}^2}\right), \end{aligned} \tag{26}$$

which is an increasing function of p_α and p_β, and a decreasing function of p_b.

Suppose the nodes in one module are connected by α links and those in the other are connected by β links with the same probability p_m, i.e., $p_\alpha = p_\beta = p_m$, as shown in Fig. 5 , and denote by $\xi = p_b/(p_m + p_b)$, the critical point which then can be briefly written by

$$\rho_c = \frac{\mu_1}{\mu_2} \frac{(1+\xi)^2}{4\xi}, \tag{27}$$

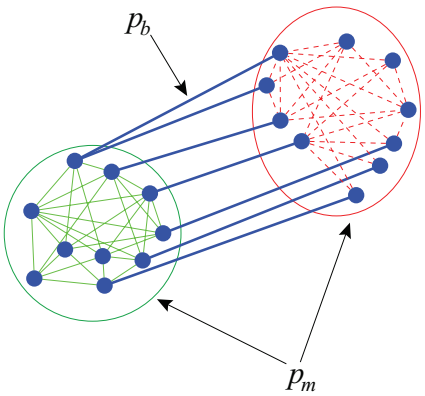

FIG. 5. (Color online) A duplex RD network with two modules. All nodes in one module are connected by α links and those in the other module are connected by β links, with the same probability p_m, while the nodes of different modules are connected by common links with probability p_b.

which is statistically equal to the first term of Eq. (13) by considering that $\xi = p_b/(p_m + p_b) = L/U = \theta$. Similarly, the assumption of no degree correlations in the two subnetworks cannot be held when $p_b \ll p_m$, and the critical point calculated by Eq. (27) needs to be further modified by

$$\rho_c = \frac{\mu_1}{\mu_2}\frac{(1+\xi)^2}{4\xi} + \frac{(1+\xi)\left[\sqrt{4+\mu_1^2(1-\xi)^2}-2\right]}{4\xi\mu_2}. \quad (28)$$

Generally, the critical point is a decreasing function of the ratio p_b/p_m. In other words, active particles can be significantly suppressed on an RD network with a distinct modular structure, i.e., $\rho_c \to \infty$ as $p_b/p_m \to 0$, which is supported by simulated results, as shown in Fig. 6(a). In this experiment, each network has two modules and each module has 200 nodes with $p_b = 0.05$ and $p_m = 0.05,0.1,0.2,0.3,0.4,0.5$, and we find that the critical point ρ_c increases as p_m increases. Moreover, the relationship between ρ_c and the two connection probabilities p_b and p_m for duplex modular networks including different numbers of nodes, i.e., $V = 200$, $V = 400$, and $V = 800$, are visualized by the three-dimensional graphs in Figs. 6(b)–6(d), respectively. When $p_b \geqslant p_m$, as the modular structure is vanishing, the RD process experiences a phase transition at a critical point close to $\rho_c = \mu_1/\mu_2 = 2$. In the opposite direction, when $0 < p_b \ll p_m$, as the modular structure is strengthened while the network is still connected, the critical point $\rho_c \to \infty$, as is predicted by Eq. (27) or (28). This is because, in this situation, different kinds of particles may be trapped into different modules, and thus their local ideological or chemical purities are more likely to be preserved; thus the reaction activity is weakened. Similarly, here, when the RD process on the network is relatively steady after sufficient time steps (10^6 here), it is parallelly implemented extra 10^4 time steps with the reaction rates $\mu_1 = 0.1$ and $\mu_2 = 0.05$ and the corresponding results are recorded.

Note that since the analytic result represented by Eq. (28) is also obtained by applying the mean-field theory with the assumption that the network size is sufficiently large, again it is inevitable to find similar discrepancies between simulated results and analytic predictions, when the network size is not large enough, as shown in Fig. 6(b). Similarly,

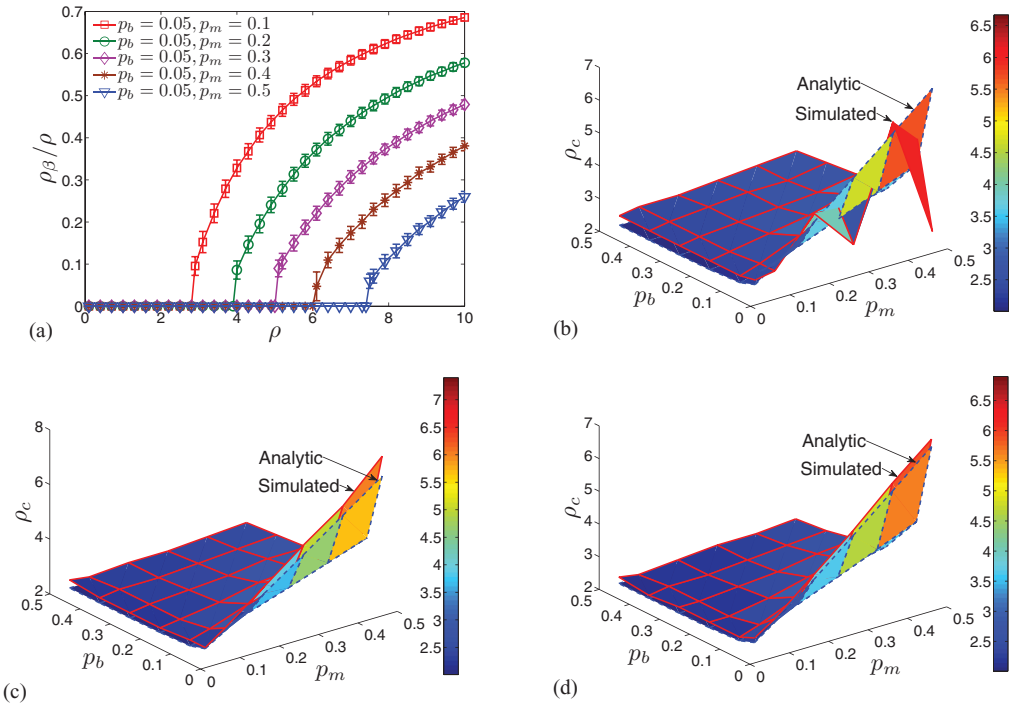

FIG. 6. (Color online) (a) Phase transitions for RD processes on duplex networks with 400 nodes consisting of two modules of the same size, for various parameters $p_b = 0.05$ and $p_m = 0.1,0.2,0.3,0.4,0.5$. (b)–(d) The analytic [calculated by Eq. (28)] and simulated critical points as functions of the inter- and intramodule connection probabilities p_b and p_m for duplex modular networks including different numbers of nodes, i.e., $V = 200$, $V = 400$, and $V = 800$, respectively.

such discrepancy will decrease gradually as the network size increases, as shown in Figs. 6(c) and 6(d). In order to provide a more quantitative result, we calculate the average relative difference ϵ between analytic and simulated critical points over all the 36 cases of different p_b and p_m, which is equal to 16.7%, 9.6%, and 6.3% for the modular duplex networks with $V = 200$, $V = 400$, and $V = 800$, respectively. In most cases, the second term of Eq. (28) is far smaller than the first term; as a result, similar relative differences can be obtained if we use Eq. (27) to calculate the critical points.

VI. CONCLUSION

This work provides a more general framework to study RD processes by considering that different types of links are used to transfer different kinds of particles in a duplex RD network. It is found that the phase transition to sustain β particles is dependent on the correlation between the two degree sequences of the simplex subnetworks, i.e., it is suppressed if the α-degree and β-degree sequences are positively correlated while extended if they are negatively correlated. In particular, when there is no correlation between the two degree sequences, it is predicted that the phase transition is equal to μ_1/μ_2, independent of the network structure. The reason is that, in such a situation, statistically, the number of β particles flowing into the nodes with certain α degree is equal to that of β particles flowing out of these nodes, so that the effect of diffusion on the RD process can be neglected. As special cases, when α or β particles do not diffuse in the network, the same results are obtained, as already revealed by Colliza *et al.* [19].

These results imply the effect of isolation on suppressing epidemics. In fact, today local governments always alert their people not to travel to the cities or the countries with higher infection rate when an epidemic has broken out. This process is simulated by our model where β particles are trapped into one module while α particles are trapped into the other, in which case, the phase transition to generate β particles can be indeed extended and thus the epidemic can be effectively restrained. On the other hand, the results also indicate that the positive correlation between different types of traffic networks has a potential risk to stimulate the epidemic spreading. This phenomenon should be further verified in the situation where differences between links are quantitatively reflected by their abilities to transfer certain particles, that is, one type of link may transfer certain particles with a higher probability than another, rather than transferring only one kind of particle, which is more practical and belongs to our future work.

ACKNOWLEDGMENTS

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APPENDIX A: SOLUTION OF RD EQUATION ON DUPLEX NETWORKS

Each node v_i has two neighboring sets, $\pi_{\alpha,i}$ and $\pi_{\beta,i}$, in α subnetwork and β subnetwork, with $k_{\alpha,i}$ and $k_{\beta,i}$ as its α degree and β degree, respectively. Denoting the present numbers of α and β particles in node v_i by $n_{\alpha,i}$ and $n_{\beta,i}$, respectively, since both the diffusion rates of α particles and β particles are set to 1, the RD dynamics are described by

$$\frac{\partial n_{\alpha,i}}{\partial t} = -n_{\alpha,i} + \sum_{j\in\pi_{\alpha,i}} \frac{\mu_1 n_{\beta,j} + n_{\alpha,j} - \mu_2\Gamma_j}{k_{\alpha,j}}, \quad \text{(A1)}$$

$$\frac{\partial n_{\beta,i}}{\partial t} = -n_{\beta,i} + \sum_{j\in\pi_{\beta,i}} \frac{(1-\mu_1) n_{\beta,j} + \mu_2\Gamma_j}{k_{\beta,j}}, \quad \text{(A2)}$$

where the reaction kernel Γ_i takes the form of $\Gamma_i = n_{\alpha,i}n_{\beta,i}$. Suppose that there are $V(k_\alpha,k_\beta)$ nodes of α-degree k_α and β-degree k_β in the network, and denote the numbers of α and β particles by $N_\alpha(k_\alpha,k_\beta)$ and $N_\beta(k_\alpha,k_\beta)$, respectively, in these nodes. Then the densities of α and β particles in the nodes of α-degree k_α are defined by

$$\rho_\alpha(k_\alpha,k_\beta) = \frac{N_\alpha(k_\alpha,k_\beta)}{V(k_\alpha,k_\beta)}, \quad \rho_\beta(k_\alpha,k_\beta) = \frac{N_\beta(k_\alpha,k_\beta)}{V(k_\alpha,k_\beta)}. \quad \text{(A3)}$$

Denote by $P(k_\alpha,k_\beta)$ the joint distribution of α degree and β degree of the network. Then, by the mean-field theory and under the assumption of no degree correlation between linked nodes in each subnetwork, Eqs. (A1) and (A2) become

$$\frac{\partial \rho_\alpha(k_\alpha,k_\beta)}{\partial t} = -\rho_\alpha(k_\alpha,k_\beta) + \frac{k_\alpha}{\langle k_\alpha\rangle}[\mu_1\rho_\beta + \rho_\alpha - \mu_2\Omega], \quad \text{(A4)}$$

$$\frac{\partial \rho_\beta(k_\alpha,k_\beta)}{\partial t} = -\rho_\beta(k_\alpha,k_\beta) + \frac{k_\beta}{\langle k_\beta\rangle}[(1-\mu_1)\rho_\beta + \mu_2\Omega], \quad \text{(A5)}$$

with $\rho_\alpha = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\rho_\alpha(k_\alpha,k_\beta)$, $\rho_\beta = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\rho_\beta(k_\alpha,k_\beta)$, and $\Omega = \sum_{k_\alpha}\sum_{k_\beta} P(k_\alpha,k_\beta)\Omega(k_\alpha,k_\beta)$, where the reaction kernel now takes the form of $\Omega(k_\alpha,k_\beta) = \rho_\alpha(k_\alpha,k_\beta)\rho_\beta(k_\alpha,k_\beta)$.

Then the stationary states are obtained by setting $\partial_t\rho_\alpha(k_\alpha,k_\beta) = 0$ and $\partial_t\rho_\beta(k_\alpha,k_\beta) = 0$, which result in the following equations:

$$\rho_\alpha(k_\alpha,k_\beta) = \frac{k_\alpha}{\langle k_\alpha\rangle}[\mu_1\rho_\beta + \rho_\alpha - \mu_2\Omega], \quad \text{(A6)}$$

$$\rho_\beta(k_\alpha,k_\beta) = \frac{k_\beta}{\langle k_\beta\rangle}[(1-\mu_1)\rho_\beta + \mu_2\Omega]. \quad \text{(A7)}$$

Multiplying Eq. (A7) by $P(k_\alpha,k_\beta)$ and summing it over k_α and k_β, we get

$$\rho_\beta = \frac{\mu_2}{\mu_1}\Omega. \quad \text{(A8)}$$

By using Eq. (A8), Eqs. (A6) and (A7) are further simplified as

$$\rho_\alpha(k_\alpha,k_\beta) = \frac{k_\alpha}{\langle k_\alpha\rangle}\rho_\alpha, \quad \rho_\beta(k_\alpha,k_\beta) = \frac{k_\beta}{\langle k_\beta\rangle}\rho_\beta, \quad \text{(A9)}$$

with

$$\rho_\alpha = \rho - \frac{\mu_2}{\mu_1}\Omega, \quad \rho_\beta = \frac{\mu_2}{\mu_1}\Omega. \quad \text{(A10)}$$

Then, from Eqs. (A9) and (A10) and by definition of Ω, we get the average densities of α and β particles, respectively, on

the network:

$$\rho_\alpha = \frac{\mu_1}{\mu_2}\frac{\langle k_\alpha\rangle\langle k_\beta\rangle}{\langle k_\alpha k_\beta\rangle}, \quad \rho_\beta = \rho - \frac{\mu_1}{\mu_2}\frac{\langle k_\alpha\rangle\langle k_\beta\rangle}{\langle k_\alpha k_\beta\rangle}. \quad \text{(A11)}$$

Denoting the node α-degree vector and the corresponding β-degree vector of the network by $S_\alpha = [k_{\alpha,1},k_{\alpha,2},\ldots,k_{\alpha,V}]$ and $S_\beta = [k_{\beta,1},k_{\beta,2},\ldots,k_{\beta,V}]$, respectively, Eq. (A11) can be rewritten as

$$\rho_\alpha = \frac{\mu_1}{\mu_2}\frac{V\langle k_\alpha\rangle\langle k_\beta\rangle}{S_\alpha {S_\beta}^T}, \quad \rho_\beta = \rho - \frac{\mu_1}{\mu_2}\frac{V\langle k_\alpha\rangle\langle k_\beta\rangle}{S_\alpha {S_\beta}^T}. \quad \text{(A12)}$$

APPENDIX B: MINIMUM/MAXIMUM CRITICAL POINT OF PHASE TRANSITION

Equation (A12) suggests a critical point in the phase transition on the surviving of β particles as

$$\rho_c = \frac{\mu_1}{\mu_2}\frac{V\langle k_\alpha\rangle\langle k_\beta\rangle}{\sum_{i=1}^{V} k_{\alpha,i}k_{\beta,i}}. \quad \text{(B1)}$$

When the diffusion and the reaction rates as well as the node α-degree and β-degree sequences of the RD network are provided, Eq. (B1) gives a formula to adjust the critical point ρ_c through determining the corresponding relationship between the two degree sequences.

Given an ordered vector $X = [x_1,x_2,\ldots,x_V]$, satisfying $x_i \geqslant x_j$ if $i > j$, denote two different types of operators $\Phi_h(Y)$ and $\Psi_h(Y)$ with $h = 1,\ldots,V$, for another disordered vector $Y = [y_1,y_2,\ldots,y_V]$. The objective is to find the maximum or minimum element in $Y_h = [y_1,y_2,\ldots,y_{V-h+1}]$ and change its place with y_{V-h+1} in the vector Y. Denote the maximum and the minimum elements in the vector Y by y_i and y_j, respectively. Since $y_i \geqslant y_V \geqslant y_j$ and $x_V \geqslant x_i,x_j$, it must be satisfied that

$$\begin{aligned} y_i x_V + y_V x_i &\geqslant x_i y_i + x_V y_V, \\ y_j x_V + y_V x_j &\leqslant x_j y_j + x_V y_V, \end{aligned} \quad \text{(B2)}$$

which together implies that

$$X[\Phi_1(Y)]^T \geqslant XY^T \geqslant X[\Psi_1(Y)]^T. \quad \text{(B3)}$$

Denote $Y_\Phi = \Phi_V[\Phi_{V-1}(\ldots\Phi_0(Y)\ldots)]$ and $Y_\Psi = \Psi_V[\Psi_{V-1}(\ldots\Psi_0(Y)\ldots)]$. Then, for any disordered vector $Y = [y_1,y_2,\ldots,y_V]$, we have

$$XY_\Phi{}^T \geqslant XY^T \geqslant XY_\Psi{}^T, \quad \text{(B4)}$$

where every two elements y_i and y_j in Y_Φ satisfy $y_i \geqslant y_j$ if $i > j$, and the elements are inversely ordered in Y_Φ. Given the node α-degree vector $S_\alpha = [k_{\alpha,1},k_{\alpha,2},\ldots,k_{\alpha,V}]$ and the corresponding β-degree vector $S_\beta = [k_{\beta,1},k_{\beta,2},\ldots,k_{\beta,V}]$ of the RD network, the above inequalities imply that the critical point ρ_c is minimized when the two vectors are in the same order, i.e., $k_{\alpha,i} \geqslant k_{\alpha,j}$ if and only if $k_{\beta,i} \geqslant k_{\beta,j}$, while it is maximized when one of them is inversely ordered.

Suppose that only the average α degree and β degree of the RD network are provided, and the node α-degree vector $S_\alpha = [k_{\alpha,1},k_{\alpha,2},\ldots,k_{\alpha,V}]$ and the node β-degree vector $S_\beta = [k_{\beta,1},k_{\beta,2},\ldots,k_{\beta,V}]$ are in the same order and satisfy that $L_\alpha \leqslant k_{\alpha,i} \leqslant k_{\alpha,j} \leqslant U_\alpha$ and $L_\beta \leqslant k_{\beta,i} \leqslant k_{\beta,j} \leqslant U_\beta$ if $i < j$. Without loss of generality, suppose that there are l elements equal to L_α and u elements equal to U_β in S_α, with $l + u < V - 1$. Then, it is possible to further increase the value of $S_\alpha {S_\beta}^T$ by the process: decrease the element $k_{\alpha,l+1}$ by δ and increase the element $k_{\alpha,V-u}$ by δ in S_α, with $\delta = \min(k_{\alpha,l+1} - L_\alpha, U_\alpha - k_{\alpha,V-u})$. In such a situation, it is satisfied that

$$\begin{aligned} &(k_{\alpha,l+1} - \delta)k_{\beta,l+1} + (k_{\alpha,N-u} + \delta)k_{\beta,N-u} \\ &\quad = k_{\alpha,l+1}k_{\beta,l+1} + k_{\alpha,N-u}k_{\beta,N-u} + \delta(k_{\beta,N-u} - k_{\beta,l+1}) \\ &\quad \geqslant k_{\alpha,l+1}k_{\beta,l+1} + k_{\alpha,N-u}k_{\beta,N-u}. \end{aligned} \quad \text{(B5)}$$

The process does not change the average α degree of the RD network and is not terminated until the total number of the elements L_α and U_α in the vector S_α is not smaller than $V - 1$. Similarly, the process is implemented on the vector S_β. Then the optimal S_α and S_β have the following forms:

$$S_\alpha = [L_\alpha,\ldots,L_\alpha,k_{\alpha,v},U_\alpha,\ldots,U_\alpha], \quad \text{(B6)}$$

$$S_\beta = [L_\beta,\ldots,L_\beta,k_{\beta,\omega},U_\beta,\ldots,U_\beta], \quad \text{(B7)}$$

with $L_\alpha \leqslant k_{\alpha,v} \leqslant U_\alpha$ and $L_\beta \leqslant k_{\beta,v} \leqslant U_\beta$. In the same manner, the value of $S_\alpha {S_\beta}^T$ is minimized, i.e., the critical point is maximized, by inversely ordering S_α or S_β. For simplicity, suppose that $L_\alpha = L_\beta = L$, $U_\alpha = U_\beta = U$, and $\langle k_\alpha\rangle = \langle k_\beta\rangle = R$, if $L < R \leqslant (L+U)/2$, the number of U in each degree vector is close to κV with κ being calculated by

$$\kappa = \frac{R-L}{U-L} \leqslant \frac{1}{2}. \quad \text{(B8)}$$

Then the minimum and maximum critical points can be estimated by

$$\rho_{c,\min} \approx \frac{\mu_1}{\mu_2}\frac{R^2}{\kappa U^2 + (1-\kappa)L^2}, \quad \text{(B9)}$$

$$\rho_{c,\max} \approx \frac{\mu_1}{\mu_2}\frac{L^2}{2\kappa UL + (1-2\kappa)L^2}. \quad \text{(B10)}$$

If $L \ll U$, we have $\kappa \approx (R-L)/U$, then Eqs. (B9) and (B10) are further simplified by

$$\rho_{c,\min} \approx \frac{\mu_1}{\mu_2}\frac{R^2}{U(R-L)}, \quad \text{(B11)}$$

$$\begin{aligned} \rho_{c,\max} &\approx \frac{\mu_1}{\mu_2}\frac{UR^2}{2UL(R-L) + (U - 2(R-L))L^2} \\ &= \frac{\mu_1}{\mu_2}\frac{UR^2}{2L(R-L)(U-L) + UL^2} \\ &\approx \frac{\mu_1}{\mu_2}\frac{R^2}{2LR - L^2}. \end{aligned} \quad \text{(B12)}$$

APPENDIX C: SOLUTION OF RD EQUATION ON DIRECTED NETWORKS WITH TWO NODES

It should be noted that the assumption of no degree correlations in the two subnetworks may not be satisfied when the lower bound of degree L is especially small, i.e., close to 1; thus the results obtained by the mean-field theory should be modified. In such a situation, both the α subnetwork and β subnetwork may possess a similar quasistar structure. When their sequences are positively correlated, they are merged into a unique quasistar RD network, while if the two degree

sequences are negatively correlated, they are merged into an RD network having two modules with the nodes in each module being connected by the same type of links and the nodes of different modules being connected by common links. Since both the networks are composed of two classes of nodes with those of the same class being completely identical, it is equivalent to study the RD processes on networks with only two coupled supernodes (a supernode represents a class of identical nodes in the initial RD network) in the two cases as shown in Figs. 2(e) and 2(f), respectively, in the paper.

1. Case I: $\eta_{\alpha,B\to A}=\eta_{\beta,B\to A}=1$, $\eta_{\alpha,A\to B}=\eta_{\beta,A\to B}=\theta$

In this case, the stationary states of the RD system with only two coupled nodes are described as in the following equations:

$$\begin{aligned}\rho_{\alpha,A}&=(1-\theta)[\mu_1\rho_{\beta,A}+\rho_{\alpha,A}-\mu_2\Omega_A]\\&\quad+[\mu_1\rho_{\beta,B}+\rho_{\alpha,B}-\mu_2\Omega_B],\end{aligned}\tag{C1}$$

$$\rho_{\alpha,B}=\theta[\mu_1\rho_{\beta,A}+\rho_{\alpha,A}-\mu_2\Omega_A],\tag{C2}$$

$$\begin{aligned}\rho_{\beta,A}&=(1-\theta)[(1-\mu_1)\rho_{\beta,A}+\mu_2\Omega_A]\\&\quad+[(1-\mu_1)\rho_{\beta,B}+\mu_2\Omega_B],\end{aligned}\tag{C3}$$

$$\rho_{\beta,B}=\theta[(1-\mu_1)\rho_{\beta,A}+\mu_2\Omega_A],\tag{C4}$$

with $\Omega_A=\rho_{\alpha,A}\rho_{\beta,A}$ and $\Omega_B=\rho_{\alpha,B}\rho_{\beta,B}$. By substituting the values of $\rho_{\alpha,B}$ and $\rho_{\beta,B}$ into Eq. (C1) and rearranging terms, we get

$$\big(\mu_1+\mu_1\theta-\mu_1^2\theta\big)\rho_{\beta,A}-(\mu_2-\mu_1\mu_2\theta)\Omega_A-\mu_2\Omega_B=0.\tag{C5}$$

Since only the phase transition to sustain β particles is concerned, when $\rho_\beta\to 0$, Eq. (C2) is simplified as

$$\rho_{\alpha,B}=\theta\rho_{\alpha,A}.\tag{C6}$$

Then the value of Ω_B is calculated by

$$\begin{aligned}\Omega_B&=\rho_{\alpha,B}\times\rho_{\beta,B}\\&=\theta[\mu_1\rho_{\beta,A}+\rho_{\alpha,A}-\mu_2\Omega_A]\times\theta[(1-\mu_1)\rho_{\beta,A}+\mu_2\Omega_A]\\&=(1-\mu_1)\theta^2\Omega_A-\mu_2\theta^2\rho_{\alpha,A}\Omega_A.\end{aligned}\tag{C7}$$

Substituting Eq. (C7) into Eq. (C5) and considering that $\Omega_A=\rho_{\alpha,A}\rho_{\alpha,A}$, we have

$$\begin{aligned}&\mu_2^2\theta^2\rho_{\alpha,A}^2+\mu_2(1+\theta-\mu_1\theta-\mu_1\theta^2)\rho_{\alpha,A}\\&\quad-\mu_1(1+\theta-\mu_1\theta)=0.\end{aligned}\tag{C8}$$

By solving this quadratic equation, we get the density of α particles in node A:

$$\rho_{\alpha,A}=\frac{\mu_1}{\mu_2}.\tag{C9}$$

From Eqs. (C6) and (C9), we get

$$\rho_\alpha=\frac{\mu_1}{\mu_2}\frac{1+\theta}{2},\quad \rho_\beta=\rho-\frac{\mu_1}{\mu_2}\frac{1+\theta}{2}.\tag{C10}$$

2. Case II: $\eta_{\alpha,B\to A}=\eta_{\beta,A\to B}=1$, $\eta_{\alpha,A\to B}=\eta_{\beta,B\to A}=\theta$

In this case, the stationary states of the RD system are described by

$$\begin{aligned}\rho_{\alpha,A}&=(1-\theta)[\mu_1\rho_{\beta,A}+\rho_{\alpha,A}-\mu_2\Omega_A]\\&\quad+[\mu_1\rho_{\beta,B}+\rho_{\alpha,B}-\mu_2\Omega_B],\end{aligned}\tag{C11}$$

$$\rho_{\alpha,B}=\theta[\mu_1\rho_{\beta,A}+\rho_{\alpha,A}-\mu_2\Omega_A],\tag{C12}$$

$$\rho_{\beta,A}=\theta[(1-\mu_1)\rho_{\beta,B}+\mu_2\Omega_B],\tag{C13}$$

$$\begin{aligned}\rho_{\beta,B}&=(1-\theta)[(1-\mu_1)\rho_{\beta,B}+\mu_2\Omega_B]\\&\quad+[(1-\mu_1)\rho_{\beta,A}+\mu_2\Omega_A].\end{aligned}\tag{C14}$$

Considering $\rho_\beta\to 0$, Eq. (C12) is simplified as

$$\rho_{\alpha,A}=\frac{1}{\theta}\rho_{\alpha,B}.\tag{C15}$$

Then, from Eqs. (C13) and (C15), the value of Ω_A is calculated by

$$\begin{aligned}\Omega_A&=\rho_{\alpha,A}\times\rho_{\beta,A}\\&=\frac{1}{\theta}\rho_{\alpha,B}\times\theta[(1-\mu_1)\rho_{\beta,B}+\mu_2\Omega_B]\\&=(1-\mu_1)\Omega_B+\mu_2\rho_{\alpha,B}\Omega_B.\end{aligned}\tag{C16}$$

Substituting Eqs. (C13) and (C16) into Eq. (C14) and considering that $\Omega_B=\rho_{\alpha,B}\rho_{\alpha,B}$, we have

$$\mu_2^2\rho_{\alpha,B}^2+\mu_2(2-\mu_1-\mu_1\theta)\rho_{\alpha,B}-\mu_1(1+\theta-\mu_1\theta)=0.\tag{C17}$$

By solving this quadratic equation, we get the density of α particles in node B:

$$\rho_{\alpha,B}=\frac{\sqrt{4+\mu_1^2(1-\theta)^2}+\mu_1(1+\theta)-2}{2\mu_2}.\tag{C18}$$

From Eqs. (C15) and (C18), we get

$$\begin{aligned}\rho_\alpha&=\left(1+\frac{1}{\theta}\right)\left[\frac{\sqrt{4+\mu_1^2(1-\theta)^2}+\mu_1(1+\theta)-2}{4\mu_2}\right],\\\rho_\beta&=\rho-\left(1+\frac{1}{\theta}\right)\left[\frac{\sqrt{4+\mu_1^2(1-\theta)^2}+\mu_1(1+\theta)-2}{4\mu_2}\right].\end{aligned}\tag{C19}$$

[1] B. A. Grzybowski, *Chemistry in Motion: Reaction-Diffusion Systems for Micro- and Nanotechnology* (Wiley, Chichester, West Sussex, 2009).

[2] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations* (Wiley, Chichester, West Sussex, 2003).

[3] V. Méndez, S. Fedotov, and W. Horsthemke, *Reaction-Transport Systems: Mesoscopic Foundations, Fronts, and Spatial Instabilities* (Springer-Verlag, Heidelberg, 2010).

[4] A. M. Turing, Philos. Trans. R. Soc. B **237**, 37 (1952).

[5] S. Kondo and T. Miura, Science **329**, 1616 (2010).

[6] S. Volpert and S. Petrovskii, Physics of Life Reviews **6**, 267 (2009).
[7] V. Colizza, A. Barrat, M. Barthélemy, and A. Vespignani, Proc. Natl. Acad. Sci. USA **103**, 2015 (2006).
[8] P. Wang, M. C. González, C. A. Hidalgo, and A.-L. Barabási, Science **324**, 1071 (2009).
[9] L. Lizana, Z. Konkoli, B. Bauer, A. Jesorka, and O. Orwar, Annu. Rev. Phys. Chem. **60**, 449 (2009).
[10] A.-L. Barabási and Z. N. Oltvai, Nature Reviews Genetics **5**, 101 (2004).
[11] M. S. Goldman, Neuron **61**, 621 (2009).
[12] D. Centola, Science **329**, 1194 (2010).
[13] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[14] R. Albert, J. Cell Sci. **118**, 4947 (2005).
[15] Q. Xuan, F. Du, and T.-J. Wu, Chaos **19**, 023101 (2009).
[16] L. K. Gallos and P. Argyrakis, Phys. Rev. Lett. **92**, 138301 (2004).
[17] S. Weber and M. Porto, Phys. Rev. E **74**, 046108 (2006).
[18] A. Baronchelli, M. Catanzaro, and R. Pastor-Satorras, Phys. Rev. E **78**, 016111 (2008).
[19] V. Colizza, R. Pastor-Satorras, and A. Vespignani, Nat. Phys. **3**, 276 (2007).
[20] L.-Q. Fang, S. J. de Vlas, D. Feng, S. Liang, Y.-F. Xu, J.-P. Zhou, J. H. Richardus, and W.-C. Cao, Trop. Med. Int. Health **14**, 14 (2009).
[21] M. Askar, O. Mohr, T. Eckmanns, G. Krause, and G. Poggensee, Euro Surveill. **17**, 20195 (2012).
[22] J. A. Stegeman, A. R. W. Elbers, A. Bouma, and M. C. M. de Jong, Epidemiol. Infect. **128**, 285 (2002).
[23] R. M. Anderson, R. M. May, and B. Anderson, *Infectious Diseases of Humans: Dynamics and Control* (Oxford University Press, Oxford, 1992).
[24] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E **62**, R5875 (2000).
[25] S. Meloni, A. Arenas, and Y. Moreno, Proc. Natl. Acad. Sci. USA **106**, 16897 (2009).
[26] R. Parshani, S. Carmi, and S. Havlin, Phys. Rev. Lett. **104**, 258701 (2010).
[27] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[28] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1999).
[29] M. Catanzaro, M. Boguñá, and R. Pastor-Satorras, Phys. Rev. E **71**, 027103 (2005).
[30] G. Palla, I. Derényi, I. Farkas, and T. Vicsek, Nature (London) **435**, 814 (2005).
[31] M. Rosvall and C. T. Bergstrom, Proc. Natl. Acad. Sci. USA **105**, 1118 (2008).
[32] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon, Science **298**, 824 (2002).
[33] X. Xu, J. Zhang, and M. Small, Proc. Natl. Acad. Sci. USA **105**, 19601 (2008).